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# Confining Flux Tubes in a Current Algebra Approach

Katherine M. Benson and Maha Saadi

*Department of Physics*

*University of California, San Diego*

*La Jolla, California 92093*

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## Abstract

We describe flux tubes and their interactions in a low energy sigma model induced by  $SU(N_f) \rightarrow SO(N_f)$  flavor symmetry breaking in  $SO(N_c)$  QCD. Unlike standard QCD, this model allows gauge confinement to manifest itself in the low energy theory, which has unscreened spinor color sources and global  $Z_2$  flux tubes. We construct the flux tubes and show how they mediate the confinement of spinor sources. We further examine the flux tubes' quantum stability, spectrum and interactions. We find that flux tubes are Alice strings, despite ambiguities in defining parallel transport. Furthermore, twisted loops of flux tube support skyrmion number, just as gauged Alice strings form loops that support monopole charge. This model, while phenomenologically nonviable, thus affords a perspective on both the dynamics of confinement and on subtleties which arise for global Alice strings.

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## I. INTRODUCTION AND MODEL

Nonlinear sigma models have had great success in describing low energy QCD phenomenology. However, they have not captured the hallmark feature of QCD: confinement, where the potential between  $q\bar{q}$  pairs grows linearly with separation. Such linear potentials can arise as tension-carrying flux tubes — between  $q$  and  $\bar{q}$ , for Yang-Mills QCD, or between any unscreened sources which persist in low energy effective theories. The conventional Skyrme model, induced by the global flavor symmetry breaking  $SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_{diag}$ , supports neither unscreened sources nor flux tubes. Global flux tubes are classified by  $\pi_2(G/H)$ , due to their constant vev at spatial infinity. This necessarily vanishes whenever the vacuum manifold  $G/H$  is itself a Lie group, as occurs in the Skyrme model.

Witten [1] noted that an  $SO(N_c)$  gauge theory of QCD induces a different Skyrme model, whose topology *can* support flux tubes. This theory has  $N_f$  lefthanded quarks  $q_L$ , which transform as (real) fundamentals under color  $SO(N_c)$ .  $SO(N_c)$  has no gauge anomaly, and thus requires no right-handed quarks; however, its  $U(1)_B$  anomaly breaks baryon number to  $Z_2$ . The high energy theory thus has the full symmetry group  $Z_2 \times SO(N_c) \times SU(N_f)$ . At low energies, flavor  $SU(N_f)$  breaks to its  $SO(N_f)$  subgroup, due to formation of a quark condensate  $\langle q_{Li}^\alpha \Sigma_{ij} q_{Lj}^\alpha \rangle$  (where  $\alpha$  and  $i$  are color and flavor indices respectively). This condensate interacts with quark excitations in the theory, inducing an effective Majorana mass for the quarks

$$\mathcal{L}_m = -\mu \left( \overline{q_L} \Sigma q_{Lc} + \overline{q_{Lc}} \Sigma^\dagger q_L \right) , \quad (1.1)$$

which displays the  $SU(N_f) \rightarrow SO(N_f)$  flavor symmetry breaking explicitly. The Goldstone modes  $\Sigma$  are described by a  $SU(N_f)/SO(N_f)$  nonlinear sigma model, with skyrmions ( $\pi_3(G/H) = Z_2$ , for  $N_f \geq 4$ ) and flux tubes ( $\pi_2(G/H) = Z_2$ , for  $N_f \geq 3$ ). This defect classification makes physical sense: for baryons, identified with antibaryons because real quarks are identified with antiquarks; and for flux tubes, whose  $Z_2$  structure emerges as a

response to external spinor sources, which can be screened by fundamental quarks only in even combinations.

In this paper, we elaborate on our recent work with Manohar [2], constructing the flux tubes in this theory and showing that their interactions with skyrmions and spinor sources obey heuristic expectations. Confinement of spinor sources in an  $SO(N_c)$  gauge theory can thus manifest itself in the low energy sigma model, through relic phenomenology in the presence of unscreened sources.

We organize our results as follows: section II derives the unique flux tube form with minimal energy; section III examines its classical stability and dynamics; section IV its quantum stability and spectrum; and section V its interactions and relationships with other fundamental objects in the theory. The early sections reveal that minimal flux tubes lie in a planar subspace of the vacuum manifold  $SU(N_f)/SO(N_f)$ ; while section V discusses why our flux tubes are Alice strings, despite ambiguities in defining parallel transport around them; why they carry skyrmion number when twisted; and how they can be viewed as mediating the confinement of spinor sources.

## II. FINDING NONTRIVIAL FLUX TUBES

To construct topological defects, we seek configurations where the condensate  $\Sigma$  varies spatially in both nonsingular and nontrivial ways. Nonsingular variations assume a form dictated by the transformation properties of  $\Sigma$ . Demonstrating their nontriviality, however, is complicated in our modified Skyrme model. The difficulty stems from the same source as the flux tubes themselves: the fact that the global vacuum manifold  $G/H$  is not itself a Lie group, but only a quotient space. Thus, unlike skyrmions in the standard Skyrme model, winding numbers for  $\Sigma$  cannot be obtained from a group structure on  $G/H$  alone, as indices dependent only on  $\Sigma$ . Instead they must be determined by homotopy arguments from the embedding of  $\Sigma$  in  $G$ .

The  $SU(2)_L \times SU(2)_R$  Skyrme model presents a simpler case. Here symmetry group

transformations  $(g_L, g_R)$  act on  $G$  by left multiplication and leave  $\Sigma$  invariant under the diagonal subgroup.  $\Sigma$  can thus be constructed from underlying group transformations:  $\Sigma = g_L g_R^\dagger$ , transforming as  $\Sigma \rightarrow L \Sigma R^\dagger$  under the group element  $(L, R)$ . When  $\Sigma = \mathbb{1}$ ,  $L = R$  gives the diagonal unbroken subgroup. For  $\Sigma \neq \mathbb{1}$ , however, the embedding  $H \subset G$  changes and  $L = R = g$  rotates  $\Sigma$  on the vacuum manifold,  $\Sigma \rightarrow g \Sigma g^\dagger$ . Using these transformation laws, and imposing spherical symmetry to minimize energy, we may write an arbitrary  $\pi_3$  defect as  $\Sigma = L(\Omega) \Sigma_o(r) R^\dagger(\Omega)$ . We choose our basis to diagonalize  $\Sigma_o(r) = \exp \{iF(r)\tau_z\}$ , with  $F(r \rightarrow \infty) = 0$ .  $\Sigma$  then has finite energy only when  $L = R$ , keeping  $\Sigma$  constant at spatial infinity, and is singular at the origin unless  $F(0) = n\pi$ .

Establishing this nonsingular form  $\Sigma = g(\Omega) \exp \{iF(r)\tau_z\} g^\dagger(\Omega)$  requires only the quotient space structure  $G/H = (SU(2)_L \times SU(2)_R) / SU(2)_{diag}$ . We require the full Lie group structure of  $G/H$ , however, to show that a particular  $g(\Omega)$  induces nontrivial  $\pi_3$ , or equivalently, to reduce the Pontryagin index to a function of  $\Sigma$ . Since  $\pi_3(G/H) = \pi_3(SU(2)) = \mathbb{Z}$ , the homotopy index  $\pi_3$  simply measures the number of times  $\Sigma(r, \Omega)$  covers  $SU(2)$ . This is easily determined by group integration or inspection. For the skyrmion,  $g(\Omega) = \exp \{-i\theta\tau_z/2\} \exp \{-i\phi\tau_y/2\}$  rotates  $\tau_z$  to an arbitrary Lie algebra element  $\hat{r} \cdot \hat{\tau}$ . Thus  $\Sigma = \exp \{iF(r)\hat{r} \cdot \hat{\tau}\}$  consists of the exponentiation of the full Lie algebra from 0 to  $n\pi$  — by definition, covering the Lie group  $n$  times, for a  $\pi_3$  winding  $n$ .

Our model yields a nonsingular ansatz for  $\Sigma$  as directly as the standard one.  $G = SU(N_f)$ , acting on itself by left multiplication, leaves  $\Sigma$  invariant under the orthogonal subgroup. We can thus write  $\Sigma$  as  $gg^T$ , transforming as  $\Sigma \rightarrow a \Sigma a^T$  under the group element  $a$ . When  $\Sigma = \mathbb{1}$ ,  $a \in SO(N_f)$  gives the unbroken subgroup  $H$ ; elsewhere, the embedding  $H \subset G$  is parallel transported to  $gHg^\dagger$ . Again, a cylindrically symmetric defect can be written as an  $r$ -dependent vev, with angle-dependent group rotation:  $\Sigma = g(\theta) \Sigma_o(r) g^T(\theta)$ . Choosing  $\Sigma_o = \mathbb{1}$  at spatial infinity restricts  $g(\theta)$  to lie in the unbroken  $SO(n)$  subgroup, and to commute with  $\Sigma_o(r = 0)$ , if  $\Sigma$  is to be nonsingular with finite energy.

We can further characterize  $\Sigma$  in terms of  $su(N_f)$  basis generators. These specify rotations in all 2-dimensional subplanes  $(jk)$ , and are usually taken as follows (with Cartan

norm  $\text{tr } T_a T_b = \frac{1}{2} \delta_{ab}$ ). The rank  $N_f - 1$  Cartan subalgebra has as basis diagonal matrices  $T_d$ ,

$$T_d = (2d(d+1))^{-1/2} \text{ diag}(1, \dots, 1, -d, 0, \dots, 0), \quad (2.1)$$

with ones in the first  $d$  entries and  $d = 1$  to  $N_f - 1$ . We can overspecify this basis, sacrificing the Cartan norm, by the set  $\{\frac{1}{2} \tau_{z(jk)}\}$  in all subplanes of the  $N_f$  dimensional vector space. Off-diagonal generators  $\{\frac{1}{2} \tau_{x(jk)}, \frac{1}{2} \tau_{y(jk)}\}$  complete the  $su(N_f)$  basis. Flavor symmetry breaking divides the basis into two sets: antisymmetric matrices  $\{T_h\} \equiv \{\frac{1}{2} \tau_{y(jk)}\}$ , generating  $H = SO(N_f)$ ; and symmetric matrices  $\{T_b\} \equiv \{\frac{1}{2} \tau_{x(jk)}, \frac{1}{2} \tau_{z(jk)}\}$ , generating no symmetries.  $\Sigma_o(r)$ , a unitary symmetric matrix, can be written  $\exp\{iF_b(r)T_b\}$ . Thus  $\Sigma$  assumes the most general nonsingular form  $\Sigma = h(\theta) \exp\{iF_b(r)T_b\} h^{-1}(\theta)$ , for  $h(\theta) \in H$  and  $F_b(r)$  ranging from zero at infinity to  $2\pi n \delta_{bb'}$  (for some fixed direction  $b'$ ) at the origin.

Having obtained nonsingular configurations  $\Sigma$ , we must demonstrate their nontriviality. Unlike the conventional Skyrme model, this requires an understanding of how  $\Sigma = gg^T$  arises from the underlying group mapping  $g$ . For flux tubes, we must construct  $\Sigma$  from the exact sequence

$$\begin{aligned} \pi_2(SU(N_f)) = 0 &\rightarrow \pi_2(SU(N_f)/SO(N_f)) \\ &\rightarrow \pi_1(SO(N_f)) = Z_2 \rightarrow \pi_1(SU(N_f)) = 0 \quad . \end{aligned}$$

That is,  $gg^T$  gives a nontrivial  $\Sigma$  only if  $g$  corresponds to some mapping from the plane to  $SU(N_f)$ , with boundary values in the  $SO(N_f)$  subgroup. Furthermore, when parametrized as a family of loops,  $g$  must start at the identity and end on a nontrivial loop in  $SO(N_f)$ . Taking  $(\alpha \in [0, 2\pi], \beta \in [0, \pi])$  as our coordinates on the plane, these criteria become

$$g(\alpha, \beta) = \begin{cases} \mathbb{1} & \text{when } \alpha = 0, \alpha = 2\pi, \text{ or } \beta = 0 \\ h^2(\alpha) & \text{when } \beta = \pi, \end{cases} \quad (2.2)$$

where  $h^2(\alpha)$  is a nontrivial loop in  $SO(N_f)$ . Such nontrivial loops can be written  $h^2(\alpha) = \exp\{i\alpha 2n_h T_h\}$ , where  $T_h$  is the set  $\{\frac{1}{2} \tau_{y(jk)}\}$  introduced above and  $n_h T_h$  generates rotations

in a single plane. (Of course, these loops can be deformed, but deformations from geodesic form lengthen the loop and induce additional gradient energy in  $\Sigma$ . We discard them, to focus on mappings that produce minimal gradient energy.)

We now show that the embeddings  $g(\alpha, \beta)$  have minimal gradient energy only when they induce flux tubes  $\Sigma$  of a unique form. We show this by imposing consistency and minimal energy conditions on the most general trivialization  $g(\alpha, \beta)$ . We then relate the topological coordinates  $(\alpha, \beta)$  to cylindrical coordinates  $(r, \theta)$ , to obtain the physical flux tube  $\Sigma(r, \theta)$ . Finally we consider the low-lying deformations of  $\Sigma(r, \theta)$  which can be favored by potential energy terms. We thus obtain a family of non-trivial flux tubes, among whom a minimal representative is selected dynamically.

We construct the most general trivialization  $g(\alpha, \beta)$  as follows. Start, at  $\beta = 0$ , by left multiplying  $h(\alpha)$  by its inverse:  $g(\beta = 0) = h^{-1}(\alpha) h(\alpha) = \mathbb{1}$ . As  $\beta$  varies, allow the left multiplier  $h^{-1}(\alpha)$  to vary over the full group  $G$ :  $g(\alpha, \beta) = g_1(\beta) h^{-1}(\alpha) g_2(\beta) h(\alpha)$ . The  $\alpha = 0$  condition and uniqueness of the inverse imply  $g_2(\beta) = g_1^{-1}(\beta)$ . To probe structure on  $G/H$  (where  $a \rightarrow aa^T$  identifies cosets  $\{gH\}$ ), write  $g_1(\beta) = b(\beta) \tilde{h}(\beta)$ , for  $\tilde{h}(\beta) \in H$  and  $b$  generated by broken generators  $T_b$ . (Specifically, since  $g_1(\beta) g_1^T(\beta) = \exp \{iG_b(\beta)T_b\}$ , we can write  $g_1(\beta) = \exp \{iG_b(\beta)T_b/2\} \tilde{h}(\beta)$ .) Thus we may write the most general trivialization of the loop  $h^2(\alpha)$  in  $G$ :

$$g(\alpha, \beta) = b(\beta) \tilde{h}(\beta) h^{-1}(\alpha) \tilde{h}^{-1}(\beta) b^{-1}(\beta) h(\alpha). \quad (2.3)$$

This induces minimal energy only in geodesic form

$$\begin{aligned} b(\beta) &= \exp \{il\beta n_b T_b\} \\ \tilde{h}(\beta) &= \exp \{im\beta \tilde{n}_h T_h\} \\ h(\alpha) &= \exp \{i\alpha n_h T_h\}, \end{aligned} \quad (2.4)$$

where  $n_b$  and  $n_h, \tilde{n}_h$  are unit vectors over the range of  $T_b, T_h$  respectively.

We now show that consistency reduces the distinct choices of  $b, h$ , and  $\tilde{h}$ . For simplicity, take  $\tilde{h} = \mathbb{1}$  at first. Choose a basis in  $su(N_f)$  so that  $n_b T_b = \frac{1}{2} \tau_z(12)$ . Then, under

conjugation by  $b(\beta)$ ,  $2n_h T_h = n_{(jk)} \tau_{y(jk)}$  rotates as follows:

$$\begin{aligned} b(\beta) 2n_h T_h b^{-1}(\beta) &= n_{(12)} \left( \cos l\beta \tau_{y(12)} + \sin l\beta \tau_{x(12)} \right) + \sum_{2 < j < k} n_{(jk)} \tau_{y(jk)} \\ &\quad + \sum_{\substack{j=1,2 \\ k>2}} n_{(jk)} \left( \cos(l\beta/2) \tau_{y(jk)} - (-1)^j \sin(l\beta/2) \tau_{x(jk)} \right). \end{aligned} \quad (2.5)$$

To obtain  $g(\alpha, \beta = \pi) = h^2(\alpha)$ , this conjugated generator must give  $-2n_h T_h$  at  $\beta = \pi$ . This can occur for only two lowest winding possibilities:  $l$  can be 1, with  $n_{(jk)} = e_{(12)}$  vanishing outside the (12) plane; or  $l$  can be 2, with  $n_{(jk)}$  describing a plane that intersects (12) in a single line. The second possibility is further constrained by the boundary condition at  $\alpha = 2\pi$ , which requires  $h(2\pi)$  to commute with  $n_b T_b$ . This occurs only when  $[n_b T_b, (n_h T_h)^2]$  vanishes — that is, when the plane  $n_{(jk)}$  intersects (12) along a coordinate axis. Thus only three distinct candidates arise for the pair  $b, h$ :  $l = 1, n_{(ij)} = e_{(12)}$ ;  $l = 2, n_{(ij)} = e_{(13)}$ ; and  $l = 2, n_{(ij)} = e_{(23)}$ .

Allowing nontrivial  $\tilde{h}(\beta)$  produces no additional flux tubes of minimal energy, as we demonstrate explicitly in an appendix. Thus the form 2.3 for the flux tubes's embedding reduces to two candidates:

$$g(\alpha, \beta) = b(\beta) h^{-1}(\alpha) b^{-1}(\beta) h(\alpha), \quad (2.6)$$

with

$$h(\alpha) = \exp \{i\alpha T_{h\square}\} \quad b(\beta) = \begin{cases} \exp \{i\beta T_{b\square}\} \\ \exp \{i2\beta T_{b\square'}\} \end{cases}, \quad (2.7)$$

where  $\square$  and  $\square'$  denote planes intersecting along a coordinate axis, and  $T_b, T_h$  are  $\frac{1}{2}\tau_z, \frac{1}{2}\tau_y$  in the indicated planes. Written in this form, we see that the two candidates are in fact the same. From equation 2.5, both choices for  $b(\beta)$  induce the same rotated group element  $b(\beta) h^{-1}(\alpha) b^{-1}(\beta)$  (modulo sign redefinition of  $\beta$ ). We can thus choose a single deformation  $g(\alpha, \beta)$ , exploring only a planar  $SU(2)$  subgroup of  $SU(N_f)$ , to produce the flux tube of minimal gradient energy associated with the  $Z_2$  loop  $h^2(\alpha)$ . Note that  $g$  indeed gives a  $Z_2$  object, with  $g(\alpha, \beta = 2\pi) = \mathbb{1}$ . (This  $Z_2$  structure appears in the  $\beta \rightarrow -\beta$  equivalences above, as well).

From this form we construct the flux tube  $\Sigma = gg^T$  :

$$\Sigma = b(\beta) h^{-1}(\alpha) b^{-2}(\beta) h(\alpha) b(\beta). \quad (2.8)$$

This gives

$$\Sigma(\alpha, \beta) = \mathbb{1} - 2 \sin^2(\alpha/2) \sin^2 \beta \mathbb{1}_\square + i \sin^2(\alpha/2) \sin(2\beta) \tau_{z\square} - i \sin \alpha \sin \beta \tau_{x\square}, \quad (2.9)$$

where  $\mathbb{1}_\square$  gives the identity in the plane  $\square$  and vanishes outside it, and  $\mathbb{1}$  is the usual  $SU(N_f)$  identity. Of course, this form for  $\Sigma$  can deform while remaining a nontrivial flux tube. In particular, for  $\Sigma$  on  $G/H$ , the variables  $\alpha$  and  $\beta$  give coordinates on the physical plane  $R^2$ , identified to  $S^2$  by the condition  $\Sigma \rightarrow \mathbb{1}$  at spatial infinity (*i. e.* on the boundary  $\alpha \in \{0, 2\pi\}; \beta \in \{0, \pi\}$ ). We can deform  $\Sigma$  to a radially symmetric form by identifying  $r = \infty$  with this boundary, and  $r = 0$  with the center  $\alpha = \pi, \beta = \pi/2$  of the deformation  $g$  producing  $\Sigma$ . This yields

$$\begin{aligned} \Sigma(r, \theta) &= h(\theta) b(r) h^{-1}(\theta), \quad \text{for} \\ h(\theta) &= \exp\{i\theta T_{h\square}\}, \quad b(r) = \exp\{iF(r) T_{b\square}\}, \end{aligned} \quad (2.10)$$

with  $T_{h\square}, T_{b\square}$  as in our definition 2.7 for  $g(\alpha, \beta)$ .

We derive this radially symmetric form for  $\Sigma$  and fix  $F(r = 0)$  as follows. We expand equation 2.10 to obtain

$$\Sigma(r, \theta) = \mathbb{1} + (\cos(F/2) - 1) \mathbb{1}_\square + i \sin(F/2) (\cos \theta \tau_{z\square} - \sin \theta \tau_{x\square}). \quad (2.11)$$

which can be identified with equation 2.9, term by term. This gives an undeformed relation between the topological description  $g(\alpha, \beta)$  and the spatial form  $F(r), \theta$ :

$$\begin{aligned} F(r) &= 4 \tan^{-1} \left( [\sin(\alpha/2) \sin \beta]^{-2} - 1 \right)^{-1/2} \\ \theta &= 2 \tan^{-1} \left( \cot(\alpha/2) \left[ \cos \beta + \sin \beta \left( [\sin(\alpha/2) \sin \beta]^{-2} - 1 \right)^{1/2} \right]^{-1} \right), \end{aligned} \quad (2.12)$$

where  $\tan^{-1}$  gives values in a single period  $[-\pi/2, \pi/2]$ . The arguments to  $\tan^{-1}$  range from 0 to  $\pm\infty$  for  $\theta$ , and from 0 to  $\infty$  for  $F(r)$ . Thus  $\theta$  covers a range  $[-\pi, \pi]$ , suitable for a

polar angle, and  $F(r)$  ranges from 0 to  $2\pi$ .  $F(r)$  assumes the extremes of its range at the boundary  $r = \infty$ , where  $F(r) = 0$ , and the center  $r = 0$ , where  $F(r) = 2\pi$ . These boundary conditions comprise the only relevant feature of the mapping, the homotopy invariant  $F(r = 0) - F(r = \infty)$ . They establish boundary conditions for any single-winding flux tube of form 2.10:  $F(0) = 2\pi, F(r \rightarrow \infty) = 0$ .

Finally, we note that while form 2.10 for  $\Sigma$  indeed minimizes gradient energy, potential terms for  $\Sigma$  — like a quark mass term — can favor a vev other than  $\exp(iF(r) T_{b\square})$ . Potential terms consistent with unbroken  $SO(N_f)$  symmetry give a vev we can always diagonalize. We thus consider the extending  $\Sigma$  so that  $F(r) T_{b\square} \rightarrow F_d(r) T_d$ , where  $F_1 \equiv F$  and  $T_d$  varies over the Cartan subalgebra generators 2.1. The nonplanar  $T_{d>1}$  commute with both  $T_{b\square}$  and  $T_{h\square}$ ; that is, they commute with the full embedding  $g(\alpha, \beta)$ . We may thus obtain our extension by the simple modification  $g(\alpha, \beta) \rightarrow g(\alpha, \beta) \exp\{iF_{d>1}(r) T_{d>1}/2\}$ . Taking  $F_{d>1}(r)$  to vanish as  $r \rightarrow \infty$ , this does not affect the behavior of  $g$  on the boundary — hence it leaves the topology unchanged. It changes the resulting flux tube only by the desired overall multiplication  $\Sigma \rightarrow \Sigma \exp\{iF_{d>1}(r) T_{d>1}\}$ . Thus the true minimum energy flux tube is one of a family of nontrivial configurations:

$$\begin{aligned} \Sigma(r, \theta) &= h(\theta) b(r) h^{-1}(\theta), \quad \text{with} \\ h(\theta) &= \exp\{i\theta T_{h\square}\}, \quad b(r) = \exp\{iF_d(r) T_d\}, \\ \text{and } F_1(r = 0) &= 2\pi; \quad F_d(r \rightarrow \infty) = 0. \end{aligned} \tag{2.13}$$

Which of these candidates is realized remains a question of dynamics.

### III. FLUX TUBE STABILITY AND DYNAMICS

Studying flux tube dynamics begins with the question of stability. As minimal model for the Goldstone field  $\Sigma$ , with stable skyrmions, we have the Skyrme lagrangian

$$\mathcal{L}_0 = \frac{F_\pi^2}{16} \text{tr} \partial_\mu \Sigma \partial^\mu \Sigma^\dagger + \frac{1}{32e^2} \text{tr} [\Sigma^\dagger \partial_\mu \Sigma, \Sigma^\dagger \partial_\nu \Sigma]^2. \tag{3.1}$$

Unaltered, this lagrangian implies a size instability for all flux tubes. For under the rescaling  $\Sigma(r, \theta) \rightarrow \Sigma(\lambda r, \theta)$ , the tension of a finite flux tube can always decrease. Specifically, its quadratic contribution stays invariant, while its quartic rescales by a factor  $\lambda^2$  — leading to an energy minimized when  $\lambda = 0$ . Physically, this corresponds to flux tubes which diffuse to infinite size to lower their energy.

To stabilize the flux tubes, we must consider modifications of the minimal forms for  $\Sigma$  and  $\mathcal{L}_o$ . First, we note that adding higher derivative gradient terms to  $\mathcal{L}_o$ , analogous to the Skyrme term, cannot both stabilize  $\Sigma$  and produce a positive definite Hamiltonian. This holds because such terms can be at most second order in time derivatives, and, by Lorentz symmetry,  $r$  and  $\theta$  derivatives; thus they can never give tension contributions scaling more strongly than  $\lambda^2$ . This leaves only two options for stabilizing  $\Sigma$ . First, we can consider a potential for  $\Sigma$ , giving a tension component that scales as  $\lambda^{-2}$ . Second, we can allow the flux tube to vary along its axis — giving  $\Sigma$  some  $z$ -dependence — to obtain a quadratic contribution to the energy scaling as  $\lambda^{-2}$ . The second option, implemented by exciting zero modes along the flux tube’s  $z$ -axis, has a structure paralleling that which arises in quantizing the flux tube. We thus defer a study of its dynamics until the next section on quantization, and focus on stabilization by a potential. Ultimately both approaches give similar results: both give as minimal flux tubes representatives from the family 2.13 with vanishing  $F_{d>1}(r)$ . Stabilization by  $z$ -rotation has a richer dynamical structure, however — partly because it introduces unconstrained rotational parameters into the problem.

A stabilizing potential, on the other hand, arises naturally in our theory. By giving the quarks bare Majorana masses, we explicitly break the  $SU(N_f)$  flavor symmetry of our original QCD gauge theory. This induces a pion mass term,

$$\mathcal{L}_m = \frac{F_\pi^2 m_\pi^2}{16} \text{tr} \left( \Sigma + \Sigma^\dagger - 2 \cdot \mathbb{1} \right) , \quad (3.2)$$

in the limit of degenerate quark masses. The mass term stabilizes the flux tube while affecting its dynamics in a simple way, as we discuss below.

To study dynamics, we calculate the energy density of flux tubes of the general form

2.13. The Skyrme action 3.1 gives a gradient contribution

$$\rho_0 = \frac{F_\pi^2}{16} \text{tr} \left\{ (F'_d T_d)^2 + \frac{1}{r^2} (\tilde{T}_\square^2 - [F'_d T_d, \tilde{T}_\square]^2) \right\} , \quad (3.3)$$

where

$$\tilde{T}_\square \equiv b^{-1}(r) [T_{h\square}, b(r)] . \quad (3.4)$$

and  $r$  has been rescaled into the dimensionless units  $eF_\pi r_{phys}$ . This form for  $\rho_0$  follows algebraically, using no information about the generators  $T_d$  and  $T_{h\square}$ .  $\tilde{T}_\square$  measures the noncommutativity of the radial generator  $F_d(r) T_d$  with its angular counterpart  $T_{h\square}$ . The nonplanar  $T_{d>1}$  commute with all generators  $T_{h\square}, T_d$  — making  $\tilde{T}_\square$  and  $[F'_d T_d, \tilde{T}_\square]$  independent of  $F_{d>1}$ . This has two consequences: first, nonplanar terms contribute only the positive definite sum  $\sum_{d>1} F'_d/2$  to the gradient energy density. Second, the  $\tilde{T}_\square$ -dependent terms span only a planar  $SU(2)$  subgroup, where  $T_{d=1} = \frac{1}{2}\tau_{z\square}$  and  $T_{h\square} = \frac{1}{2}\tau_{y\square}$ . They are thus calculable, giving the flux tube gradient energy density

$$\rho_0 = \frac{F_\pi^2}{16} \left\{ \frac{1}{2} \sum_d F'_d^2 + \frac{1}{r^2} (1 - \cos F_1) (1 + F'_1)^2 \right\} . \quad (3.5)$$

The pion mass term 3.2 also contributes to the energy. Form 2.13 for the flux tube yields its contribution in closed form:

$$\rho_m = \frac{m_\pi^2}{16e^2} \left\{ 2N_f - 4 \cos \left( \sum_{d>1} \omega_d/2 \right) \cos(F_1/2) - 2 \sum_{d>1} \cos \omega_d \right\} , \quad (3.6)$$

with

$$\omega_d \equiv \frac{-d F_d}{\sqrt{2d(d+1)}} + \sum_{i>d} \frac{F_i}{\sqrt{2i(i+1)}}, \quad (3.7)$$

where  $d = 2, \dots, N_f - 1$  and  $r$  has again been rescaled to  $eF_\pi r_{phys}$ .

The minima of this potential fix boundary conditions for  $\Sigma$  at spatial infinity. Such minima have  $F_1 = 2\pi m$ , consonant with our a priori boundary condition  $F_1(r \rightarrow \infty) = 0$ . To minimize with respect to the nonplanar  $\omega_d$ , we set

$$\frac{\partial \rho_m}{\partial \omega_d} \sim 2 \sin \left( \sum_{d>1} \omega_d/2 \right) \cos(F_1/2) + 2 \sin \omega_d = 0 , \quad (3.8)$$

with positive second derivatives  $\partial^2 \rho_m / \partial \omega_d^2, \partial^2 \rho_m / \partial F_1^2$ . This gives the degenerate family  $\omega_d = 2\pi j_d$ , for any integers  $j_d$  summing to the same parity as  $m$ . As the simplest representative, we take  $\omega_2 = 2m\pi$  and  $\omega_d = 0$  for  $d > 2$ . This completes our boundary conditions at infinite radius:  $F_1 \rightarrow 0, \omega_d \rightarrow 0$ , for all  $d$ ; corresponding to  $F_d \rightarrow 0$  for all  $d$ , in our original variables.

Away from spatial infinity, we must propagate these boundary conditions inward to obtain flux tubes obeying the full equations of motion. However, given our boundary conditions  $\omega_d(r \rightarrow \infty) = 0$ , these equations trivialize for the nonplanar variables  $F_{d>1}$ . They have independent gradient energy terms 3.5, which combine with the potential 3.6 to give equations of motion

$$\frac{F_\pi^2}{16} \frac{1}{r} \partial_r (r F'_d) = \frac{\partial \omega_e}{\partial F_d} \frac{\partial \rho_m}{\partial \omega_e} \quad \text{for } d > 1. \quad (3.9)$$

From equation 3.8, the vacuum values  $\omega_e = 2\pi j_e$  cause the source terms to vanish on the right hand side — even when the planar field  $F_1$  departs from its vacuum value. Thus our boundary conditions, setting  $F_d = \omega_d = 0$  at infinite radius, induce only constant solutions as we propagate equation 3.9 inward to the origin. This gives  $F_d$  that vanish identically for nonplanar  $d > 1$ .

This implies that the minimum energy flux tube for any  $N_f$  varies only over the planar  $SU(2)$  subgroup. It takes the simple form 2.10, with  $F(r) \equiv F_1(r)$ , and has energy density

$$\rho = \frac{F_\pi^2}{16} \left\{ \frac{1}{2} F'^2 + \frac{1}{r^2} (1 - \cos F) (1 + F'^2) + \lambda^2 (1 - \cos(F/2)) \right\}, \quad (3.10)$$

where  $\lambda = 2m_\pi/e F_\pi$ . This determines a nonlinear equation of motion for  $F$ :

$$\begin{aligned} \left( 1 + \frac{4}{r^2} \sin^2(F/2) \right) \frac{1}{r} \partial_r (r F') = \\ \sin(F/2) \left\{ \frac{2}{r^2} \cos(F/2) (1 - F'^2) + \frac{8F'}{r^3} \sin(F/2) + \frac{\lambda^2}{2} \right\}. \end{aligned} \quad (3.11)$$

Numerical solutions to equation 3.11 are shown in Figure 1 for different values of  $\lambda$ , including the physical  $\lambda_0 = 0.236$  ( $e = 2\pi, m_\pi = 138$  Mev and  $F_\pi = 186$  Mev). Increasing  $\lambda$  raises the flux tube's energy density while shrinking its core size. The asymptotic regimes agree with limits to equation 3.11. Inside the core,  $F$  falls linearly from its boundary value

of  $2\pi$  at the origin; outside, it scales as the hyperbolic Bessel function  $F \sim x^{-1/2} \exp\{-x\}$ , for  $x = \lambda r/2$ . The parameter  $2\lambda^{-1}$  thus sets the flux tube's core size in dimensionless units. In physical units, this gives core size  $m_\pi^{-1}$ , sensible for a bound state of Goldstone modes  $\Sigma$ . It carries tension proportional to  $F_\pi^2$  — which is proportional to the number of colors  $N_c$  in a large  $N_c$  QCD limit. This supports its interpretation as a mediator of the confining force between spinor sources, whose scales as the spinor Casimir  $N_c$ . Numerically we find tension  $4.6F_\pi^2$  when  $\lambda = \lambda_0$ . As  $\lambda$  varies, the tension varies as shown in figure 2.

## IV. QUANTIZING THE FLUX TUBE

### A. Semiclassical Zero Modes and Quantum Stability

Under quantization, the flux tube samples not only the ground state above, but also its zero modes. Recall, from section II,  $\Sigma$ 's transformation laws:  $\Sigma \rightarrow a\Sigma a^T$  under global rotations  $a$ , giving the residual symmetry  $H = SO(N_f)$  at infinite radius where  $\Sigma = \mathbb{1}$ . However, because  $\Sigma$  varies nontrivially, global  $H$  rotations are not symmetries within the core. Instead they produce distinct degenerate configurations, coincident at spatial infinity. The zero modes of  $\Sigma$  explore these configurations:

$$\Sigma(t, r, \theta) = A(t) \Sigma(r, \theta) A^{-1}(t) , \quad (4.1)$$

where  $A(t) \equiv \exp\{ieF_\pi\omega t n_h T_h\}$  rotates in  $H$  with dimensionless frequency  $\omega$ . <sup>\*</sup> These zero modes have rotational energy confined to the string core; an energy that can be made arbitrarily small, classically, by taking  $\omega \rightarrow 0$ .

Classically, we calculate this rotational energy from the Skyrme action 3.1:

$$\rho_\omega = \frac{F_\pi^2 \omega^2}{16} \text{tr} \left( \hat{T}^2 - [F'_d T_d, \hat{T}]^2 - \frac{1}{r^2} [\tilde{T}_\square, \hat{T}]^2 \right) . \quad (4.2)$$

---

<sup>\*</sup>We neglect other excitations, such as bending modes.

Here  $\tilde{T}_\square$ , from equation 3.4, measures the noncommutativity of radial and angular generators  $F_d(r)T_d$  and  $T_{h\square}$ , while

$$\hat{T} \equiv b^{-1}(r) [ h^{-1}(\theta) n_h T_h h(\theta), b(r) ] \quad (4.3)$$

measures the noncommutativity of  $A(t)$  with  $\Sigma(r, \theta)$ . We have retreated to the general form 2.13 — including nonplanar contributions — for  $\Sigma(r, \theta)$ ; thus we retain complete expressions 3.5 and 3.6 for its gradient and potential energy. We do this for two reasons: first, we must insure that the planar vacuum  $F_{d>1} = 0$  — favored by the weak pion mass term — remains minimal despite quantum fluctuations due to zero modes; and second, we wish to explore stabilization by  $z$ -rotation, where  $A = A(z)$ .

Calculating the rotational energy 4.2 is straightforward. We expand  $2n_h T_h$  as  $n_{(jk)} \tau_{y(jk)}$ ; then note that the tilde operation, defined in equation 3.4, acts on basis elements as follows:

$$2\tilde{T}_{(jk)} = \begin{pmatrix} -i(\pi_{jk} - 1) \\ i(\pi_{jk}^* - 1) \end{pmatrix}_{(jk)} = (\cos \omega_{jk} - 1) \tau_{y(jk)} + \sin \omega_{jk} \tau_{x(jk)}. \quad (4.4)$$

Here  $\pi_{jk}$  is given by

$$\pi_{jk} \equiv \exp \{i\omega_{jk}\} \equiv \exp \{iF_d(T_{d,k} - T_{d,j})\}, \quad (4.5)$$

where  $T_{d,k}$  denotes the  $k$ -th diagonal entry in the Cartan generator  $T_d$ , from equation 2.1. The commutator terms act on the relevant basis elements  $\tau_{x(jk)}$ ,  $\tau_{y(jk)}$  to give

$$\begin{aligned} [T_d, \tau_{a(jk)}] &= -i(T_{d,k} - T_{d,j}) \epsilon_{ab} \tau_{b(jk)} \quad \forall j, k \\ [2\tilde{T}_\square, \tau_{a\square'}] &= ip(\cos \omega_\square - 1) \tau_{a\square''} + i \sin \omega_\square \epsilon_{ab} \tau_{b\square''}, \end{aligned} \quad (4.6)$$

where  $\epsilon_{ab}$  is the two dimensional permutation matrix  $\epsilon_{xy}$ . The second commutator contributes only when planes  $\square$  and  $\square'$  intersect, with  $[\tau_{y\square}, \tau_{y\square'}] = ip \tau_{y\square''}$  defining  $\square''$  and  $p = \pm 1$ . Altogether, these expressions imply the rotational energy

$$\rho_\omega = \frac{F_\pi^2 \omega^2}{16} \left\{ \begin{array}{l} n_{(12)}^2 (1 - \cos F_1) (1 + F_1'^2) + \sum_{k>j>2} n_{(jk)}^2 (1 - \cos \omega_{jk}) (1 + \omega_{jk}'^2) + \\ \frac{1}{2} \sum_{k>2} (n_{(1k)}^2 + n_{(2k)}^2) \left( (1 - \cos \omega_{1k}) (1 + \omega_{1k}'^2 + \frac{1}{2r^2} (1 - \cos F_1)) \right. \\ \left. + (1 - \cos \omega_{2k}) (1 + \omega_{2k}'^2 + \frac{1}{2r^2} (1 - \cos F_1)) \right) \end{array} \right\}. \quad (4.7)$$

This form for the rotational energy depends on the planar  $F_1$ , both explicitly and through the variables  $\omega_{jk}$ . To extract its  $F_1$  dependence, we note that

$$\omega_{jk} = \omega_{k-1} - \omega_{j-1}, \quad (4.8)$$

where

$$\omega_0 \equiv -\frac{1}{2} \left( \sum_{d>1} \omega_d \right) + F_1/2, \quad \omega_1 \equiv -\frac{1}{2} \left( \sum_{d>1} \omega_d \right) - F_1/2. \quad (4.9)$$

and the nonplanar  $\omega_{d>1}$  are given by eq. 3.7. We may thus rewrite the rotational energy as

$$\rho_\omega = \frac{F_\pi^2 \omega^2}{16} \left\{ \begin{array}{l} n_{(12)}^2 (1 - \cos F_1) (1 + F_1'^2) + \sum_{k>j>2} n_{(jk)}^2 (1 - \cos \omega_{jk}) (1 + \omega_{jk}'^2) \\ + \frac{1}{2} \sum_{k>2} (n_{(1k)}^2 + n_{(2k)}^2) \begin{pmatrix} (1 + S_{k-1}'^2 + F_1'^2/4 + \frac{1}{2r^2} (1 - \cos F_1)) \\ \cdot (1 - \cos S_{k-1} \cos(F_1/2)) \\ + F_1' S_{k-1}' \sin S_{k-1} \sin(F_1/2) \end{pmatrix} \end{array} \right\}, \quad (4.10)$$

where  $S_k \equiv \omega_k + \frac{1}{2} \sum_{d>1} \omega_d$ . This depends only on  $F_1$  (explicitly) and the nonplanar  $\omega_{d>1}$  (explicitly and through  $S_{k-1}$ ,  $\omega_{jk}$ ).

Neglecting the pion mass term for now, our boundary conditions require  $F_1$  and  $\omega_{d>1}$  to approach a minimum of the potential 4.10 at spatial infinity. As in the previous section, these minima occur when  $F_1 = 2\pi m$  and  $\omega_{d>1} = 2\pi j_d$ , for any integers  $j_d$  whose sum has the same parity as  $m$ . We take as the simplest representative  $\omega_2 = 2m\pi$  and  $\omega_{d>2} = 0$ . (The specific case  $N_f = 3$  has additional degenerate vacua, non-coincident with those of the pion mass term, given by  $\omega_2 = 2m\pi/3$ ; however, these reduce to the pion mass vacua above when more families are added.) Thus our boundary condition  $F_1(r \rightarrow \infty) = 0$  induces the full boundary conditions at infinite radius:  $F_1 \rightarrow 0$ ,  $\omega_{d>1} \rightarrow 0$ ; corresponding to  $F_d \rightarrow 0$  for all  $d$ , in our original variables.

The rotational energy  $\rho_\omega$  generates spatially-dependent corrections to the equations of motion for  $F_d$ . However, these corrections trivialize for the nonplanar variables  $F_{d>1}$ , given our boundary conditions  $\omega_d = 0$ . We show this as follows: the rotational energy 4.10 introduces an additional term

$$\Delta = \frac{\partial \omega_e}{\partial F_d} \left( \frac{\partial \rho_\omega}{\partial \omega_e} - \frac{1}{r} \partial_r \left( r \frac{\partial \rho_\omega}{\partial \omega'_e} \right) \right) \quad (4.11)$$

into the right hand side of the equation of motion 3.9 for  $F_d$ . This term is generally quite complicated; however, when the nonplanar  $\omega_{d>1}$  assume their asymptotic values of zero, it simplifies. For a single  $\omega_e$  (before multiplying by the Jacobian), it becomes

$$\begin{aligned} \Delta_e &= \omega^2 \left\{ E_{e+1} + \frac{1}{2} \sum_{k>2} E_k \right\} , \quad \text{with} \\ E_k &= -F'_1 S'_{k-1} \sin(F_1/2) - \frac{1}{r} \partial_r (2r S'_{k-1}) (1 - \cos(F_1/2)) . \end{aligned} \quad (4.12)$$

Asymptotically, where  $F_1 = 0$ , all  $\Delta_e$  vanish, and the full equation of motion 3.9 is again sourceless. This gives asymptotic solutions for the nonplanar  $F_d$  which are again Bessel functions of order zero — implying, for finite energy, that  $F_{d>1}$  and all its derivatives (or  $\omega_{d>1}$  and all its derivatives) vanish at infinite radius.

However, given nonplanar terms that obey  $\omega_{d>1} = \omega'_{d>1} = \omega''_{d>1} = 0$ , the perturbations  $\Delta_e$  vanish, regardless of the planar field  $F_1(r)$ . So, as for pion mass stabilization, equation 3.9 remains sourceless as we propagate it inward to the origin, giving  $F_{d>1}$  that vanish identically.

Thus the planar vacuum  $F_{d>1} = 0$  survives quantum fluctuations due to zero modes. Its static limit retains the form 2.10, with  $F(r) \equiv F_1(r)$  varying only over a planar  $SU(2)$  subgroup. It has classical energy, from equation 4.10,

$$\rho_{tot} = \frac{F_\pi^2}{16} \left\{ \begin{array}{l} \frac{1}{2} F'^2 + (r^{-2} + \omega^2 n_\square^2) (1 - \cos F) (1 + F'^2) + \lambda^2 (1 - \cos(F/2)) \\ + \omega^2 n_{\square'}^2 (1 - \cos(F/2)) (1 + F'^2/4 + \frac{1}{2r^2} (1 - \cos F)) \end{array} \right\} \quad (4.13)$$

where

$$n_\square^2 = n_{(12)}^2 , \quad n_{\square'}^2 = \sum_{k>2} n_{(1k)}^2 + n_{(2k)}^2 , \quad (4.14)$$

describe the orientation of the zero mode rotation  $A(t)$  relative to  $\Sigma(r, \theta)$ .

For a flux tube stabilized by  $z$ -rotation, where  $A = A(z)$ , the rotational terms in  $\rho_{tot}$  deform the nonlinear equation of motion for  $F$ . Classically, we can rescale  $\omega$  to set  $n_\square^2 + n_{\square'}^2 = 1$  (a rescaling unnecessary when  $N_f = 3$ ). This gives energy as a monotonic function of  $n_\square^2$  —

which measures the component of  $\Sigma$ 's  $z$ -dependence due to slip dislocation, where  $\Sigma(z, r, \theta)$  rotates about its core in a single internal space plane  $\square$ , but starts at the  $x$ -axis with changing offset angle  $\theta_0(z)$ . Its converse,  $n_{\square'}^2$ , measures the twist dislocation in  $\Sigma(z, r, \theta)$ : the extent to which  $A(z)$  rotates the internal space plane,  $\square \rightarrow R_{ab}(z)\square$ , in which  $\Sigma(z, r, \theta)$  cycles about its core.

Numerical solutions for the  $z$ -stabilized  $F$  are shown in Figures 3 and 4. Their asymptotic behavior agrees with analytic limits: inside the core,  $F$  falls linearly from its boundary value of  $2\pi$  at the origin; while outside, it scales as the hyperbolic Bessel function  $F \sim x^{-1/2} \exp\{-x\}$ , for  $x = (\lambda^2 + \omega^2(1 + 3n_{\square}^2))^{1/2} r/2$ . That is, the flux tube becomes more compact as either  $\omega$  or  $n_{\square}$  grows. As in the mass stabilization case, shrinking core sizes correlate with growing tensions. Thus we see, from Figure 3, that the value  $n_{\square} = 0$  is favored, and twisting dislocations cost less energy than their slip-offset counterparts. Figure 4 confirms the flux tube's tendency both to shrink and to gain energy with increasing rotational frequency, and shows how significant the rotational deformation of  $F(r)$  is. Thus the tension not only acquires rotational energy terms,

$$\tau \rightarrow \tau + \frac{1}{2}\omega^2(n_{\square}^2 \Lambda_{\square} + n_{\square'}^2 \Lambda_{\square'}) ; \quad (4.15)$$

it also contains hidden rotational dependence through deformations of the ground state tension  $\tau$  and moments of inertia  $\Lambda_{\square}, \Lambda_{\square'}$ . We explore these deformations in Figure 5, showing that  $\tau$  grows linearly with  $\omega$ , due to flux tube compression. The moments of inertia instead fall rapidly, dropping by a factor of five as  $\omega$  grows from 0 to 2, before stabilizing at roughly  $\Lambda_{\square} = 10 F_{\pi}^2, \Lambda_{\square'} = 4F_{\pi}^2$  for  $\omega \geq 2$ . We show below that this range  $\omega \geq 2$  is typical of quantum rotational excitations of the flux tube; thus we are justified in neglecting rotational deformation of  $\Lambda_{\square}, \Lambda_{\square'}$  over this range.

Finally, we note that the geodesic parametrization assumed above for the rotation  $A(t)$ , while useful for discussing the flux tube's classical limit, does not restrict our analysis. Any function  $A(t)$  over  $H = SO(N_f)$  induces the energy 4.13, with  $\omega n_{(jk)}$  defined by

$$\omega n_{(jk)} \equiv -i(eF_{\pi})^{-1} \text{tr } A^{\dagger} \dot{A} \tau_{y(jk)} . \quad (4.16)$$

## B. The Quantized Spectrum

From the previous section, there are two ways to stabilize the flux tube: by exploiting the pion mass term in the Lagrangian or by  $z$ -rotation. These give the two classical solutions  $\Sigma(r, \theta)$  (equation 2.13) and  $\Sigma(r, \theta, z)$  (equation 4.1, with  $t \rightarrow z$ ). Both have rotational zero modes induced by  $A(t) \in H$ . We now quantize the spectrum of these zero modes to find the flux tube's quantum numbers and lowlying excitations [3].

Quantizing  $\Sigma(r, \theta, z)$  is a complicated task, because it involves two rotations, the  $z$  and  $t$  rotations. Furthermore, there is no physical input for the  $z$ -rotation frequency  $\omega$ . Therefore, we restrict ourselves to quantizing the mass-stabilized solution  $\Sigma(r, \theta)$ .

Equations 4.15 and 4.16 determine a two-dimensional Lagrangian for the flux tube:

$$L(\tilde{z}, t) = -\tilde{\tau} - \frac{\tilde{\Lambda}_{\square}}{2} (\text{tr } A^\dagger \dot{A} T_{h\square})^2 - \frac{\tilde{\Lambda}_{\square'}}{2} \sum_{\square'} (\text{tr } A^\dagger \dot{A} T_{h\square'})^2, \quad (4.17)$$

where  $\tilde{\tau}$  is a ground state tension and  $\tilde{z} = eF_\pi z$  measures dimensionless length along the flux tube. The moments of inertia  $\tilde{\Lambda}_{\square}$  and  $\tilde{\Lambda}_{\square'}$  come from equation 4.13:

$$\begin{aligned} \tilde{\Lambda}_{\square} &= \frac{\pi}{e^3 F_\pi} \int r dr (1 + F'^2)(1 - \cos F) \\ \tilde{\Lambda}_{\square'} &= \frac{\pi}{e^3 F_\pi} \int r dr \left(1 + F'^2/4 + \frac{1}{2r^2}(1 - \cos F)\right) (1 - \cos(F/2)) . \end{aligned} \quad (4.18)$$

We note that form 4.17 for the Lagrangian comes from only two facts: from the confinement of  $\Sigma$  to the planar  $SU(2)_{\square}$  subgroup in internal space; and from cylindrical symmetry in physical space. The skyrmion in this theory, which we discuss in the following section, shares these characteristics. Thus its one-dimensional Lagrangian  $L(t)$  also has form 4.17, with  $\tilde{\tau}$ ,  $\tilde{\Lambda}_{\square}$  and  $\tilde{\Lambda}_{\square'}$  dependent upon skyrmion dynamics.

To quantize such a Lagrangian, we must do two things: first, we must understand the transformation properties of  $\Sigma$  under the symmetries of  $L$ ; and second, we must write  $L$  in terms of invariants of the quantized Noether charges of those symmetries. The first task is facilitated by considering the most general form of  $\Sigma$  on the vacuum manifold:

$$\begin{aligned} \Sigma(x_i, t) &= A(t) e^{iF(x_i)n_b(x_i)T_b} A^{-1}(t) \\ &= e^{iF(x_i)n_b(x_i)R_{b'b}T_{b'}} , \end{aligned} \quad (4.19)$$

where  $R_{b'b}(t)$  is an orthogonal matrix encoding the isospin rotation due to  $A$ -conjugation:

$$A^\dagger T_i A = R_{ij} T_j \Rightarrow R_{ij}(t) = 2 \operatorname{tr} (A^\dagger(t) T_i A(t) T_j) . \quad (4.20)$$

While  $R_{ij}$  is defined for all  $SU(N)$  generators  $T_i$ , it breaks into block diagonal form,  $R_{hh'} \oplus R_{bb'}$ , over the space of unbroken and broken generators respectively.

For the general form  $\Sigma(x_i, t)$ , the Lagrangian 4.17 involves a general sum  $-\frac{\tilde{\Lambda}_h}{2}(\operatorname{tr} A^\dagger \dot{A} T_h)^2$ .

This Lagrangian is invariant under global  $h \in SO(N_f)$ , implemented by either left or right multiplications  $A \rightarrow hA$  or  $A \rightarrow Ah$ . For infinitesimal transformations  $h_\epsilon = 1 - 2i\epsilon_h T_h$ ,  $\Sigma(x_i, t)$  transforms as:

$$\begin{aligned} \delta_L A = (-i2\epsilon_h T_h)A &\iff \delta_L \Sigma(x_i, t) = -i\epsilon_h [2T_h, \Sigma(x_i, t)] \\ \delta_R A = A(-i2\epsilon_h T_h) &\iff \delta_R \Sigma(x_i, t) = -i\epsilon_h A [2T_h, \Sigma(x_i)] A^\dagger . \end{aligned} \quad (4.21)$$

Therefore the left transformation corresponds to  $SO(N_f)$  isospin rotation, and its Noether charge  $\mathcal{I}_h$  satisfies an  $so(N_f)$  algebra after quantization. The right transformation corresponds to  $SO(N_f)$  isospin rotation in the body fixed frame, with Noether charge  $\mathcal{I}'_h$ . To understand the physical interpretation of  $\mathcal{I}'_h$ , we calculate the commutator in the second line of equation 4.21, obtaining

$$A[2T_h, \Sigma(x_i)]A^\dagger = 2if_{hbb'}n_b\partial_{n_{b'}}\Sigma(x_i, t) . \quad (4.22)$$

For the specific cases of flux tube and skyrmion,  $n_b(x_i)$  has two special properties:  $n_b = n_{b\square}$ , lying only in an  $SU(2)$  subplane of  $SU(N_f)$ ; and  $n_b(x_i)$  is spatially axisymmetric, depending linearly on spatial direction components  $u_x$  and  $u_y$ . These properties identify planar isospin rotations with spatial  $z$ -rotations. We see this in two steps: the planarity of  $n_b$  reduces equations 4.21 and 4.22, when  $h = h\square$ , to

$$\delta_R \Sigma(r, n_b(r, u_i)) = -2\epsilon f_{h\square b\square b'\square} n_{b\square} \partial_{n_{b'\square}} \Sigma(r, n'_b(r, u_i)) , \quad (4.23)$$

which the Jacobian relating  $n_b$  to  $u_i$ , at fixed  $u_z$ , equates with spatial  $z$ -rotation :

$$\delta_R \Sigma(r, u_i) = -2\epsilon \epsilon_{zij} u_i \partial_{u_j} \Sigma(r, u_i) . \quad (4.24)$$

Thus  $\mathcal{I}'_{\square}$ , which generates planar body-centered isospin rotations  $R$ , must equal the defect's spin operator  $J_z$ , generating spatial  $z$ -rotations. Other isospin rotations, which destroy the planarity of  $n_b$ , are distinct from other spatial rotations, which change the defect's axis of axisymmetry.

Under right and left infinitesimal transformations of  $A(t)$ , the rotations  $R_{b'b}$ , which give physical coordinates for  $\Sigma(x_i, t)$  on the vacuum manifold, transform as

$$\begin{aligned}\delta_L R_{b'b} &= -2\epsilon_h f_{hb'c} R_{cb} \\ \delta_R R_{b'b} &= -2\epsilon_h R_{b'c} f_{hcb}.\end{aligned}\quad (4.25)$$

That is,  $R_{b'b}$  is left or right multiplied by  $(\tilde{h}_\epsilon)_{bc} = 1 + i\epsilon_h(2if_{hbc})$ . This shows how isospin rotation acting directly on the space of broken generators can obey an  $so(N_f)$  algebra — specifically the  $so(N_f)$  algebra which occurs as a subalgebra of  $su(N_f)$  in the adjoint representation, restricted to the block-diagonal component  $2if_{hbc}$ .

To find  $\mathcal{I}_h$  and  $\mathcal{I}'_h$ , we compute the variation of the Lagrangian under the two infinitesimal transformations:

$$\delta_L L = \dot{\epsilon}_h \mathcal{I}_h \quad \text{and} \quad \delta_R L = \dot{\epsilon}_h \mathcal{I}'_h. \quad (4.26)$$

This gives

$$\mathcal{I}'_h = i\tilde{\Lambda}_h \text{tr}(A^\dagger \dot{A} T_h) \quad (\text{no sum}), \quad \text{and} \quad \mathcal{I}_h = R_{hh'} \mathcal{I}'_{h'}. \quad (4.27)$$

We can now write the defect Hamiltonian in terms of the physical quantized Noether charges: the isospin  $\mathcal{I}$  and angular momentum  $\mathcal{J}$ . Combining the Lagrangian 4.17 with Noether charges 4.27 yields a Hamiltonian dependent on  $\mathcal{I}_{\square}^2$  and  $\mathcal{I}_{\square'}^2$ , where  $\square'$  are planes intersecting  $\square$  in only one line. However, the defect  $\Sigma(r, u_i)$  is invariant under a subgroup  $SO(N_f - 2)$  of  $SO(N_f)$ , acting on the subspace  $\square_{\perp}$  orthogonal to  $\square$ . This implies two things:  $\mathcal{I}^2 = \mathcal{I}_{\square}^2 + \sum_{\square'} \mathcal{I}_{\square'}^2$ ; and the defect's allowed quantum states  $\mathcal{I}'$  must contain a singlet under the  $SO(N_f - 2)$  subgroup. For  $N_f > 3$ , this excludes spinor representations — constraining the skyrmions and flux tubes to be bosonic. Lastly, because  $\mathcal{I}$  is simply a rotation of  $\mathcal{I}'$ , and  $\mathcal{I}'_{\square} = \mathcal{J}_z$ , we write the Hamiltonian as

$$H(\tilde{z}, t) = \tilde{\tau} + \frac{1}{2\tilde{\Lambda}_{\square}} \mathcal{J}_z^2 + \frac{1}{2\tilde{\Lambda}_{\square'}} (\mathcal{I}^2 - \mathcal{J}_z^2), \quad (4.28)$$

subject to the constraints above:  $\mathcal{I}'_{\square} = \mathcal{J}_z$ , with bosonic defects for  $N_f > 3$ .

Using the numerical results of Section IVA, we may be more explicit about the flux tube's quantized energy levels. Relating the rotational energies of equation 4.28 to their counterparts in the tension 4.15 (for dimensionful  $z$ ) gives

$$\omega^2 = (eF_{\pi})^4 (\mathcal{J}_z^2/4\Lambda_{\square}^2 + (\mathcal{I}^2 - \mathcal{J}_z^2)/4\Lambda_{\square'}^2) \approx 4\mathcal{J}_z^2 + 25(\mathcal{I}^2 - \mathcal{J}_z^2). \quad (4.29)$$

Thus  $\omega$  assumes values  $\omega \geq 2$  for quantized zero modes. Over this range,  $\tilde{\Lambda}_{\square}$  and  $\tilde{\Lambda}_{\square'}$  vary little but  $\tilde{\tau}$  is described by the linear fit  $\tilde{\tau} = (9.0 + 3.7(\omega - 2)) F_{\pi}/e$ , as shown in Figure 5. Thus our model predicts

$$H(\tilde{z}, t) = \frac{F_{\pi}}{2\pi} \left( 1.6 + 3.7 \sqrt{4\mathcal{J}_z^2 + 25(\mathcal{I}^2 - \mathcal{J}_z^2)} + 20\mathcal{J}_z^2 + 50(\mathcal{I}^2 - \mathcal{J}_z^2) \right) \quad (4.30)$$

for the allowed excited states  $(\mathcal{I}, \mathcal{J}_z)$ , and

$$H(\tilde{z}, t) = 4.6 \frac{F_{\pi}}{2\pi} \quad (4.31)$$

for the ground state of the flux tube.

## V. FLUX TUBE INTERACTIONS

We now consider how flux tubes interact with fundamental objects in our low energy theory. First, we establish that flux tubes can be Alice strings — forcing some charges and particle wave functions to become double-valued. Like gauge theories with Alice strings and monopoles, our sigma model has twisted flux loops which form point defects. We construct these defects — the skyrmions — and demonstrate their nontrivial topology. We then consider how they interact with the flux tubes, showing that only topologically trivial combinations of two flux tubes can end on skyrmions. This suggests the physical interpretation that, while the baryons in this theory are not confined, the spinor sources which combine to form them are, with confinement mediated by the  $Z_2$  flux tubes joining them.

First we consider twisted flux tubes. We note that our flux tubes may share the defining property of Alice strings, which arise in the symmetry-breaking of certain gauge theories. [4] Alice strings have unbroken symmetries which preserve a local vev but cannot be extended globally, since they become multivalued when parallel transported around the spatially varying vev of the string. This “Alice” nature is not topologically invariant, but depends on the specific Wilson line integral producing the string’s asymptotic winding. Whether our flux tubes have this trait is similarly ambiguous: using the physical embedding 2.10 of the flux tube, we can take  $g(r, \theta) = h(\theta) b(F(r)/2)$ . This gives parallel transported unbroken generators  $g T_{h\Box'} g^{-1}$  that are double-valued in  $\theta$  for each radius. However, all generators can be made single-valued by choosing instead  $g(r, \theta) = h(\theta) b(F(r)/2) h(\theta)$ , which induces the same flux tube  $\Sigma$ .

Because  $\Sigma$  can be viewed as an Alice string, we might expect it to share a property of gauged Alice strings in models with monopoles: that twisted string loops support monopole charge. For gauged Alice strings, this is necessary, as monopoles alter their charge when passing through string loops; moreover, specific examples indicate that twisted Alice loops can carry a single unit of monopole charge. [5] For global Alice strings, the suggestion arises by analogy. In our case, it gains further support from the hidden gauge character of the low energy global theory. Specifically, we can recast the flavor-dependent quark mass  $\Sigma$  in the usual way, as an interaction between flavor gauge fields  $g^{-1}(r, \theta) \partial_\mu g(r, \theta)$  and shifted quarks  $Q_L = g(r, \theta) q_L$ . [3] This accomplishes two goals: it equates skyrmion number with anomalous baryon current, while identifying the flux tube with a gauged string, whose Wilson line integral is  $P \exp(\oint \vec{A} \cdot d\vec{l})$ . However, the Alice ambiguity persists: choosing  $g(r, \theta)$  double-valued gives a gauge theory with Wilson loop  $U(2\pi) = \mathbb{1} - 2 \mathbb{1}_\Box$ , making both the quarks  $Q_L$  and the generators  $T_{h\Box'}$  double-valued. A single-valued choice for  $g(r, \theta)$  produces instead single-valued quarks and generators, with Wilson loop  $U(2\pi) = \mathbb{1}$ .

This ambiguity can be resolved physically, by considering adiabatic transport of quarks around the flux tube. [6] Under such transport, the quarks remain in their mass eigenstates. These are governed by two terms: a flavor-independent bare Majorana mass  $M$  (breaking

$SU(N_f) \rightarrow SO(N_f)$  explicitly) and the flavor-dependent Majorana mass  $\mu\Sigma$ , where  $\mu$  is the  $SU(N_f)$ -breaking vev. The two determine mass eigenstates that are double-valued in  $\theta$ . However, the states have mass splitting  $\Delta m^2 = -2iM(\mu - \mu^*)\sin(F/2)$ , where we have chosen  $M$  real. Thus the quarks are degenerate, and unaffected by transport around the string, unless the vev  $\mu$  is misaligned in phase with the bare mass  $M$ . In that case, quarks at finite radius have double-valued wave functions and Aharonov-Bohm scatter off the flux tube, which we identify as an Alice string. <sup>†</sup>

Motivated by its gauge theoretic counterpart, we calculate the skyrmion number of a twisted flux tube:

$$\Sigma(z, r, \theta) = A(z) \Sigma(r, \theta) A^{-1}(z) , \quad (5.1)$$

where  $A(z) = \exp(iz \ln_h T_h) \in H$ . Imposing  $2\pi$ -periodicity in  $z$  constrains  $l$ : for planar  $n_h = n_{\square}$ ,  $l$  must be integral; for nonplanar  $n_h$ , it must be even. Thus any twisted flux loop can deform to the planar flux loop  $\Sigma(z, r, \theta) = h(\theta + lz) b(F(r)) h^{-1}(\theta + lz)$ , which takes values only in the two-flavor subspace  $SU(2)/SO(2) \sim S^2$ . This simplification allows us to identify the loop's  $\pi_3$  index with its Hopf number, *i.e.* the linking number between any two fibers of constant  $\Sigma$  in physical space. As discussed in [7], this linking number is precisely  $l$  — the number of times a nontrivial fiber  $\Sigma_0$  twists around the loop's core, which has  $\Sigma = -\mathbb{1}_{\square}$ . Thus flux loops with an  $l = 1$  planar twist form fundamental skyrmions; flux loops with nonplanar twist have  $l > 1$ , which is trivial for  $N_f > 3$ .

A nicer parametrization of the skyrmion stems from the exact sequence

<sup>†</sup>We consider here quarks whose trajectories remain well-separated from degeneracies at the origin; that is, for which  $r^2 \gg m_i^{-2}$ , where  $m_i$  are the quark mass eigenvalues. Thus we rely on an asymptotic regime inside the flux tube, in which  $m_i^{-1} \ll m_{\pi}^{-1}$ . The flux tube background affects general quark wave functions more subtly. Since the mass eigenstates depend explicitly on  $\theta$ , off-diagonal  $\hat{\theta} \cdot \hat{\tau}$  terms appear in the quark equations of motion, mixing mass eigenstates at small radius. This causes corrections to maximal Aharonov-Bohm scattering, much as in [6].

$$\pi_3(SO(N_f)) \rightarrow \pi_3(SU(N_f)) \rightarrow \pi_3(SU(N_f)/SO(N_f)) \rightarrow \pi_2(SO(N_f)) = 0 .$$

This implies that fundamental skyrmions in the theory can be constructed from the fundamental skyrmions

$$g(r, \hat{u}) = \exp(iF_s(r) \hat{u}_i T_{i\square}) \quad (5.2)$$

in  $SU(N_f)$ . Here  $r$  and  $\hat{u}$  are the radius and unit direction vector in 3-space, and  $F_s(r)$  approaches  $2\pi$  at  $r = 0$  and zero at  $r = \infty$ . This determines an axisymmetric skyrmion

$$\begin{aligned} \Sigma_s = \mathbb{1} - 2 \sin^2(F_s/2) (1 - u_z^2) \mathbb{1}_\square + 2i \sin(F_s/2) \cdot \\ (\cos(F_s/2) (u_y \tau_{x\square} + u_x \tau_{z\square}) + \sin(F_s/2) u_z (-u_x \tau_{x\square} + u_y \tau_{z\square})) , \end{aligned} \quad (5.3)$$

after a global spatial rotation fixing the  $z$ -axis as the axis of spatial axisymmetry. In the  $xy$ -plane, this gives

$$\Sigma_s(z = 0) = \mathbb{1} + (\cos F_s - 1) \mathbb{1}_\square + i \sin F_s (u_y \tau_{x\square} + u_x \tau_{z\square}) . \quad (5.4)$$

Comparing with equation 2.11 for the flux tube allows us to identify the angular winding  $n_b(\theta) T_{b\square}$  of a flux tube with that of a skyrmion in the  $xy$ -plane. Thus, if their radial boundary conditions coincided, we could deform the lower hemisphere of the skyrmion into a flux tube. However, because both  $F$  and  $F_s$  vary from 0 to  $2\pi$  over this plane, the skyrmion cannot end in a single flux tube. The boundary conditions instead allow the skyrmion to join only to flux tube configurations where  $F(r)$  ranges from 0 to  $4\pi$  — that is, configurations with two flux tubes, deformable to the trivial configuration. Thus skyrmions cannot be confined in this theory.

However, objects which combine to form skyrmions can interact with the flux tubes. Such “half-skyrmions” could arise as external spinor sources in the underlying theory. They should be confined, as fundamentals cannot screen them. As mappings on  $G/H$ , they appear in our theory precisely as half-skyrmions, that is, as objects of the form 5.3 with  $F_s(r)$  ranging from 0 to  $\pi$ . Such objects are not defects in the conventional sense, since they have linearly divergent energy — just like an unscreened point source. Their boundary

conditions allow them to join to their opposite winding counterparts via single flux tubes, confining their linearly divergent energy to a length scale set by the tube length. We thus see that confinement of sources in an  $SO(N_c)$  gauge theory can induce a relic phenomenology, which persists in the low energy Skyrme model.

## ACKNOWLEDGMENTS

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## APPENDIX:

In this appendix, we show that trivializations 2.3 of the loop  $h^2(\alpha)$  produce minimal flux tubes only when  $\tilde{h}(\beta) = \mathbb{1}$ . We do this by generalizing the analysis following equation 2.4 for arbitrary  $\tilde{h}(\beta)$  — only to find that consistency and energy considerations restore the restriction  $\tilde{h}(\beta) = \mathbb{1}$ .

Note that conjugation by  $\tilde{h}(\beta)$  rotates  $2T_h$  in  $so(N_f)$ :

$$n_{(jk)}(\beta) \tau_{y(jk)} \rightarrow u_{(jk)}(\beta) \tau_{y(jk)}, \quad \text{where } u_{(jk)}(\beta) \equiv \tilde{h}_{(jp)}(\beta) n_{(pq)} \tilde{h}_{(kq)}(\beta). \quad (\text{A1})$$

Since  $\tilde{h}(\beta)$  is real and orthogonal,  $u_{(jk)}(\beta)$  is just a real unit vector, like  $n_{(jk)}$ .  $b(\beta)$  then conjugates this rotated generator, giving Equation 2.5 with the substitution  $n_{(jk)} \rightarrow u_{(jk)}(\beta)$ . This fully conjugated generator must become  $-2T_h$  at  $\beta = \pi$ , to produce the loop  $g(\alpha, \beta = \pi) = h^2(\alpha)$ . This occurs only if  $u_{(jk)}(\pi) = \lambda_{(jk)} n_{(jk)}$ , for each  $(jk)$ , with

$$\lambda_{(jk)} = \begin{cases} -\exp(il\pi) & \text{for } (jk) = (12) \\ -\exp(il\pi/2) & \text{for } j \leq 2, k \geq 3 \\ -1 & \text{for } 2 < j < k. \end{cases} \quad (\text{A2})$$

We note that these matching conditions depend only on a basis choice fixing  $b(\beta)$  in the  $(12)$  plane. Rotations of  $b(\beta)$  within that plane, for example taking  $T_b \rightarrow \frac{1}{2}\tau_{x(12)}$ , change the details of equation 2.5, but not the coefficients of  $\tau_{y(jk)}$  and its orthogonal generators; and these coefficients alone fix the matching A2.

These matching conditions cannot be achieved by orthogonal conjugation of an arbitrary generator  $T_h$ . A first constraint stems from the reality of  $\lambda_{(jk)}$ . This allows a nonvanishing  $n_{(jk)}$  only when the relevant exponential gives  $\pm 1$ . For  $l = 1$ , this forces  $n_{(1k)} = n_{(2k)} = 0$  for  $k > 2$  — splitting both  $h(\alpha)$  and  $\tilde{h}(\beta)$   $h(\alpha)$   $\tilde{h}^{-1}(\beta)$  into two commuting blocks. The  $2 < j, k$  block is unaffected by  $b$ -conjugation, and changes  $g$  only by an overall right multiplication, which leaves  $\Sigma = gg^T$  invariant. Thus, for  $l = 1$ ,  $T_h = \frac{1}{2}\tau_{y(12)}$  again produces all nontrivial loops  $g$ .

Equation A1 places more subtle constraints on  $T_h$ , related to consistency of the diagonal reduction  $u_{(jk)}(\pi) = \lambda_{(jk)}n_{(jk)}$ , for all  $(jk)$ . This is solved, for independent  $n_{jk}$ , only when  $\tilde{h}_{(jp)}(\pi)$  is itself diagonal:  $\tilde{h}_{(jp)}(\pi) = \lambda_j\delta_{(jp)}$ , with  $\lambda_j$  real. This constrains the matching conditions attainable by nonzero  $n_{(jk)}$  to  $\lambda_{jk} = \lambda_j\lambda_k$ . Note that our  $n_{jk}$  are effectively independent, since the relation A2 holds for all global rotations of  $n_{(jk)}$  that leave  $b(\beta)$  in the  $(12)$ -plane. Such rotations map a single generator in any of the three matching classes —  $(jk)$  coincident with  $(12);(jk)$  intersecting  $(12)$  in a single line; and  $(jk)$  disjoint from  $(12)$  — to the entire class, in arbitrary linear combinations. Thus, for  $l = 2$ , only two possibilities are consistent with the matching conditions A2. We can have nonvanishing  $n_{(jk)}$  in the cross planes  $(1k), (2k)$ , with  $k \geq 3$ . However, this precludes nonvanishing  $n_{(jk)}$  outside the cross planes:  $\lambda_{(1k)} = \lambda_{(2k)} = 1$ , for all  $k \geq 3$ , implies  $\lambda_{(jk)} = 1$  for all  $j, k$ . Thus we must set  $n_{(jk)} = 0$  outside cross planes to obey A2. Similarly, nonzero  $n_{(12)}$ , with  $\lambda_{(12)} = -1$ , implies  $\lambda_{(1k)} = -\lambda_{(2k)}$  for  $k \geq 3$  — forcing  $n_{(jk)} = 0$  in the cross planes to obey A2. The case  $l = 2$  thus produces no new candidates for  $T_h$ : either  $n_{(jk)}$  vanishes outside the planes  $(jk)$  which intersect  $(12)$  in a single line, or  $n_{(jk)} = e_{(12)}$  in the  $(12)$  plane (discarding a right multiplication  $g \rightarrow gh'$  as in the  $l = 1$  case above).

The only surviving consequence of  $\tilde{h}$ -conjugation, then, is an overall rotation  $u_{(jk)}(\pi) =$

$\pm n_{(jk)}$ . For the positive sign,  $\tilde{h}(\beta)$  commutes with  $h^{-1}(\alpha)$  at both endpoints in  $\beta$ , so choosing  $[\tilde{h}(\beta), h^{-1}(\alpha)] = 0$  for all  $\beta$  clearly minimizes gradient energy. For the negative sign, a similar reduction occurs for  $b(\beta)$ . For then  $b(\beta)$  commutes with  $\tilde{h}(\beta)$   $h^{-1}(\alpha)$   $\tilde{h}^{-1}(\beta)$  at both endpoints, and we can deform  $g$  to lie entirely in  $H$ . Thus only the case  $\tilde{h}(\beta) = \mathbb{1}$  induces a nontrivial defect  $\Sigma \in G/H$  of minimal energy.

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## FIGURES

FIG. 1. a.) Flux tube solutions  $F(r)$  for  $\lambda$  values 0.2, 0.236, 1.0, and 2.0, with mass stabilization only ( $\omega = 0$ ). The dotted line corresponds to the physical  $\lambda_0 = 0.236$ . Note that core size shrinks with increasing  $\lambda$ . b.) The above flux tubes' energy density  $\rho$ . Note that tension grows with the coupling  $\lambda$ .

FIG. 2. Dependence on stabilizing mass ( $\omega = 0$ ). Solid line shows variation of the total tension  $\tau = \int d^2r \rho$  as a function of  $\lambda$ . Dotted line shows the gradient contribution  $\tau_0$  only. The physical  $\lambda_0 = 0.236$  is marked.

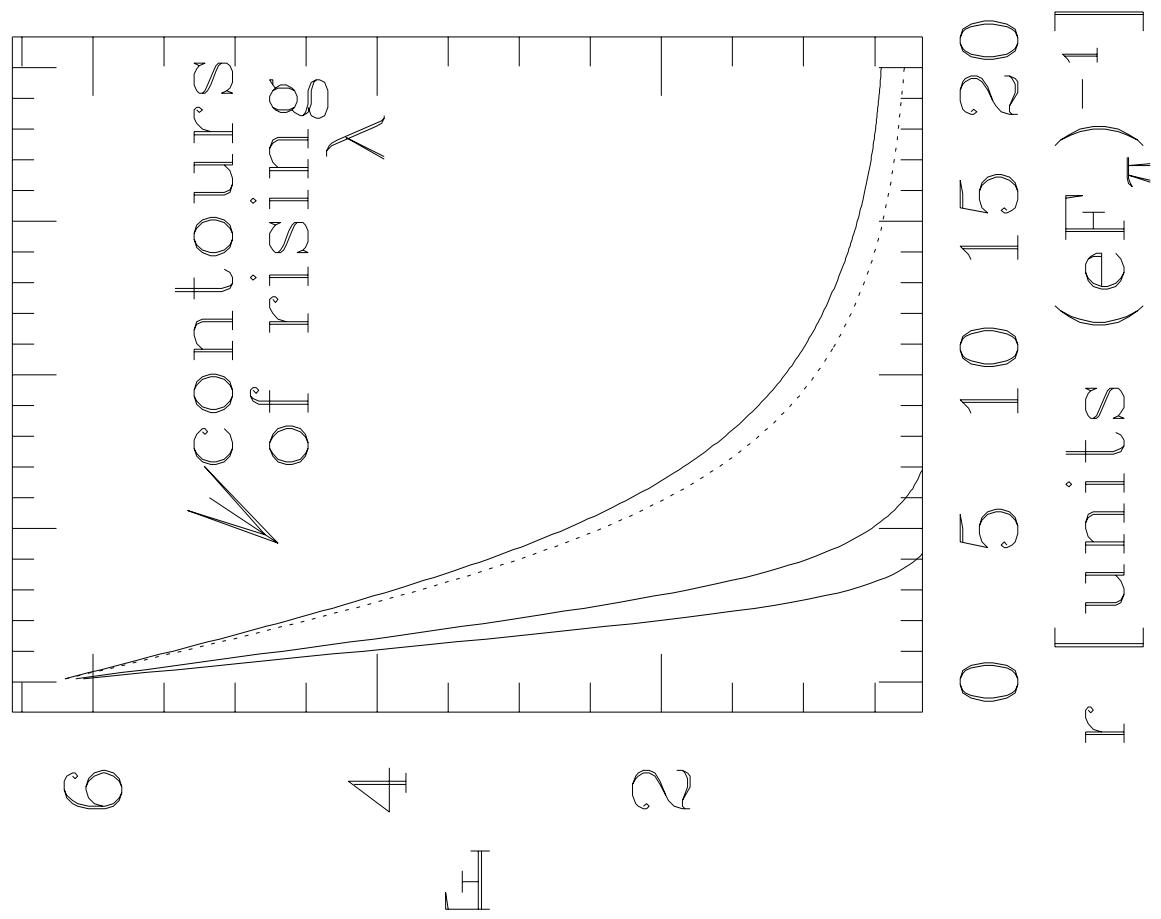
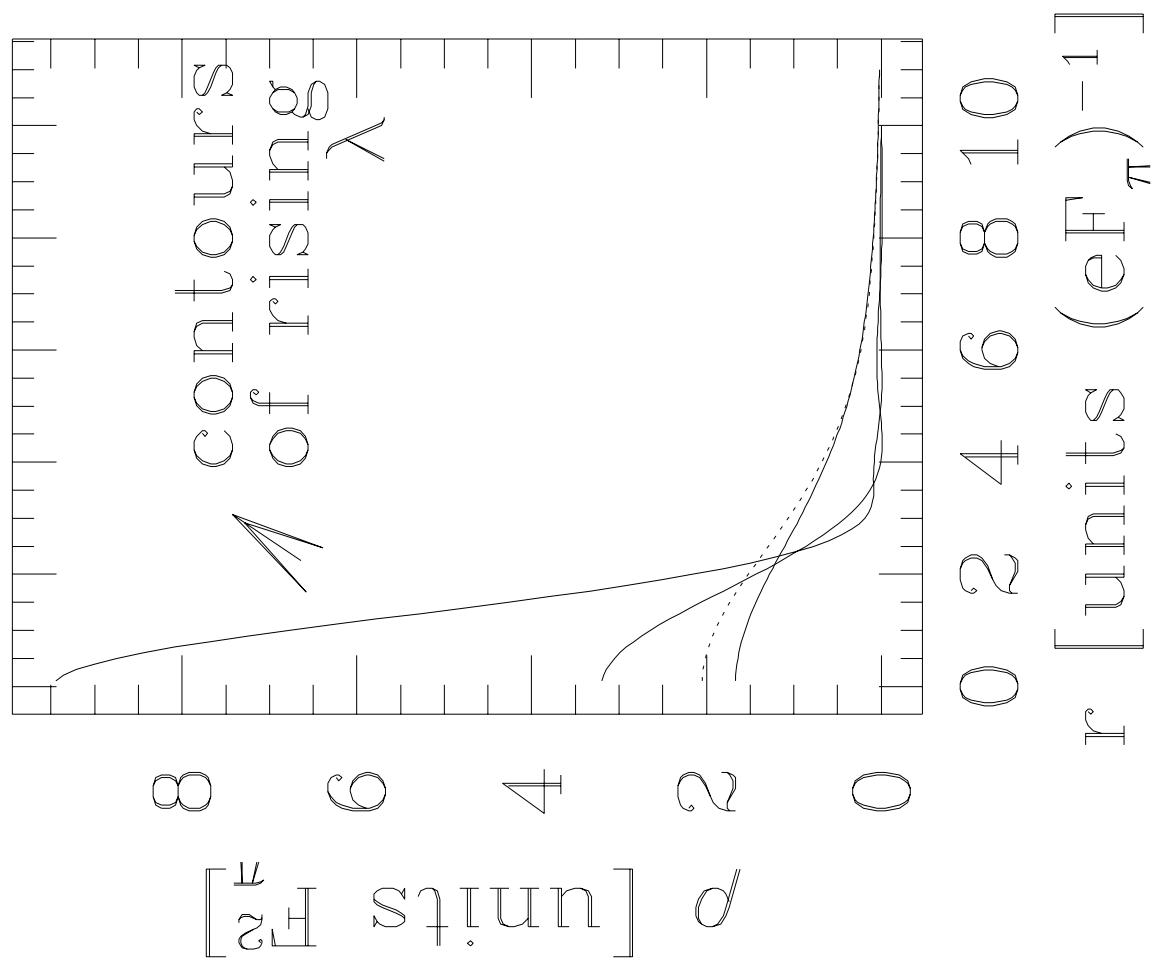
FIG. 3. Zero mode dependence on rotational direction. Solutions for  $\lambda = \lambda_0$ , with fixed frequency  $\omega = 1$ , and varying planar component  $n_{\square}$ . Note that core size shrinks and tension grows as the planar component  $n_{\square}$  increases from 0 to 1. This implies that twisting dislocations are favored over slipping ones (see text). Allowing  $\lambda$  to vanish has negligible effect on  $F$  and  $\rho_{tot}$ .

FIG. 4. The rotating mass-stabilized flux tube ( $\lambda = 0.236$ ). For  $n_{\square} = 0$ , shows the flux tube's deformation due to rotational zero modes. As for other values of  $\lambda$ , the flux tube's core size shrinks and tension grows as the rotation frequency  $\omega$  increases from 0.5 to 1.5. Rotation with a nonzero planar component  $n_{\square}$  displays the same trend, but begins with more compact and energetic flux tubes when  $\omega = 0.5$  (cf. Fig. 3).

FIG. 5. Rotational deformation, for  $n_{\square} = 0.5$ . Dotted line shows  $\lambda_0 = 0.236$ ; solid lines show the  $\lambda = 0$  and  $\lambda = 1$  cases. a) Variation of the ground state tension  $\tau$  with rotational frequency  $\omega$ . (Note that  $\tilde{\tau}$ , the tension conjugate to  $\tilde{z}$  of Section IV, is given by  $(eF_{\pi})^{-1}\tau$ ). b) and c) Variation of the planar and nonplanar moments of inertia  $\Lambda_{\square}$  and  $\Lambda_{\square'}$ . (Note that  $\tilde{\Lambda}_{\square}$  and  $\tilde{\Lambda}_{\square'}$  of section IV are given by  $4(eF_{\pi})^{-3}$  times their counterparts plotted here.)

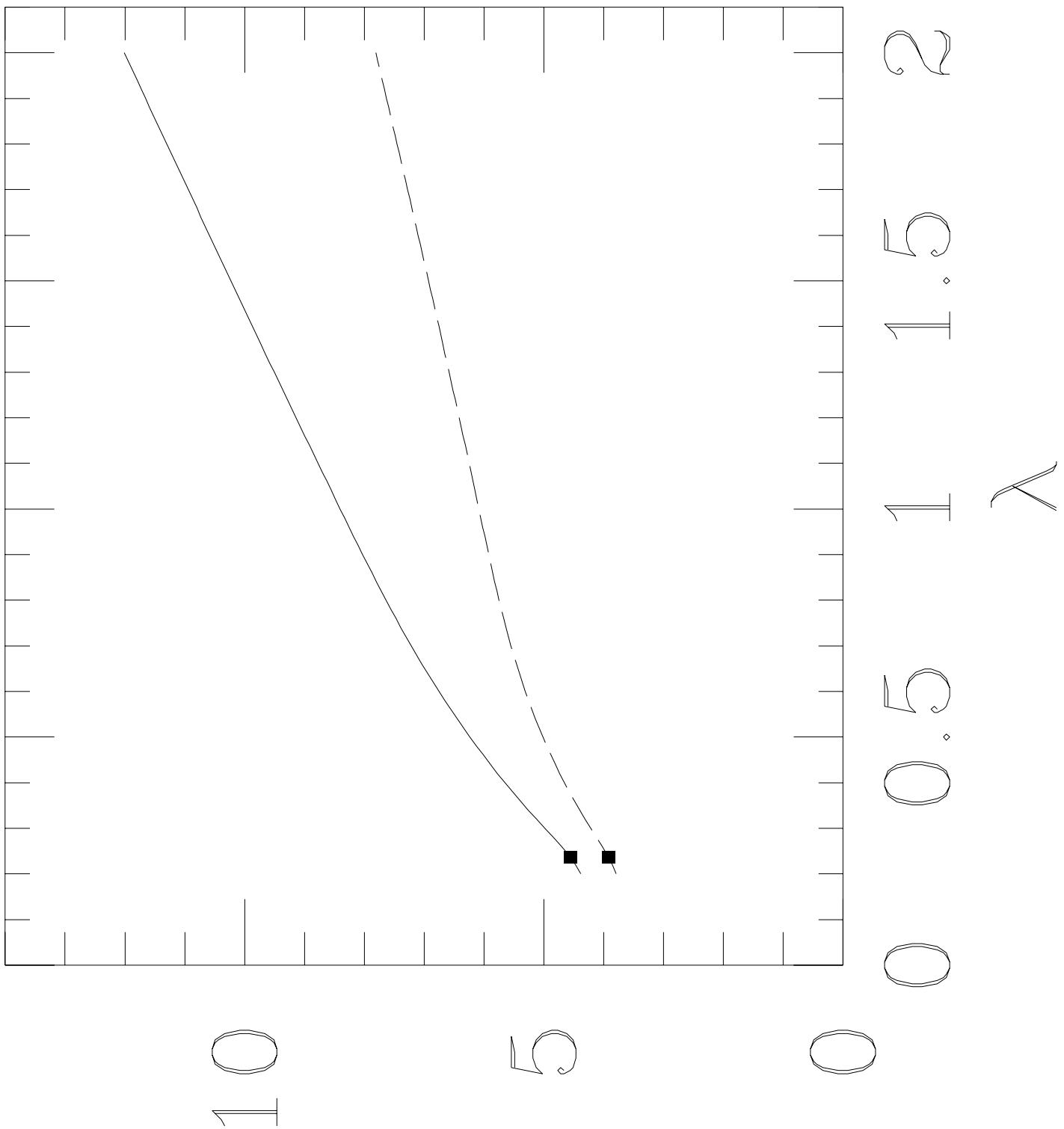
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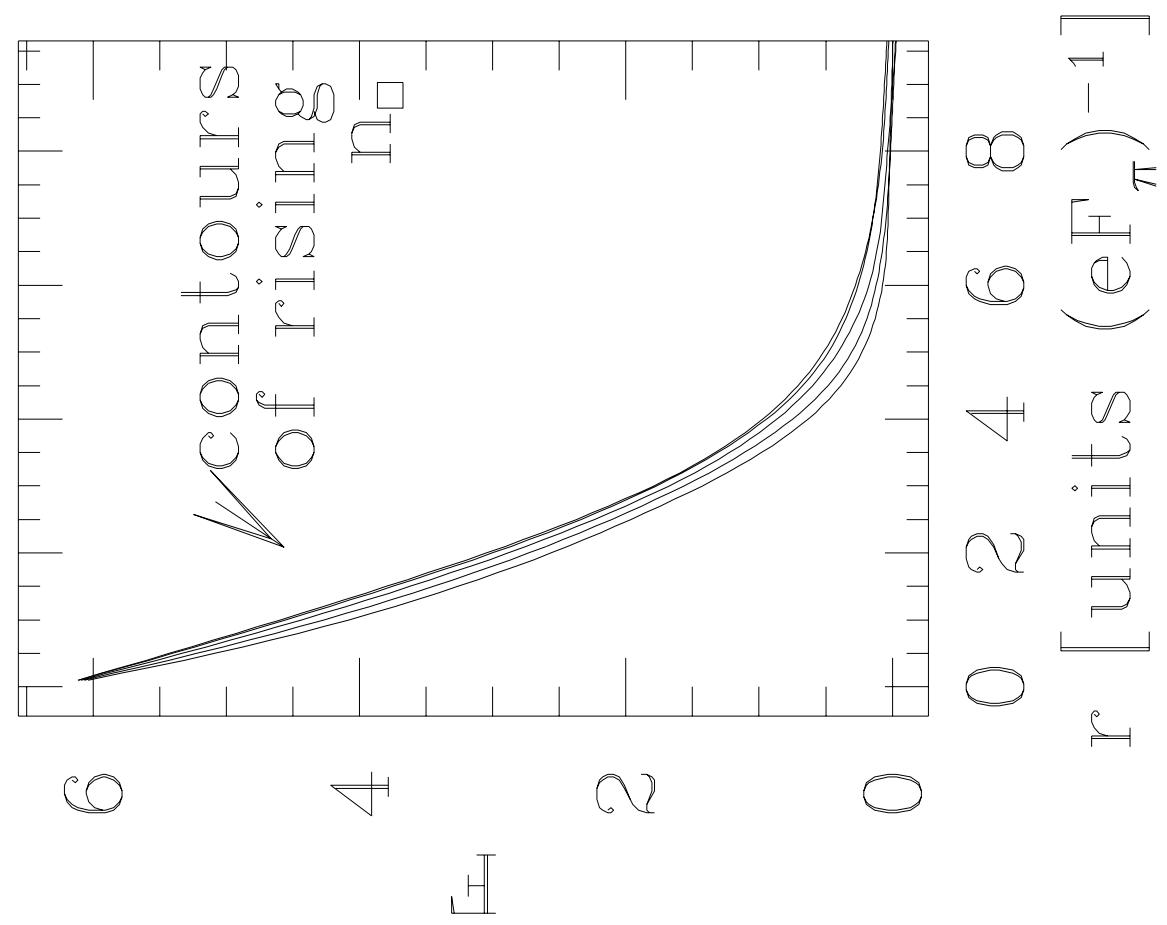
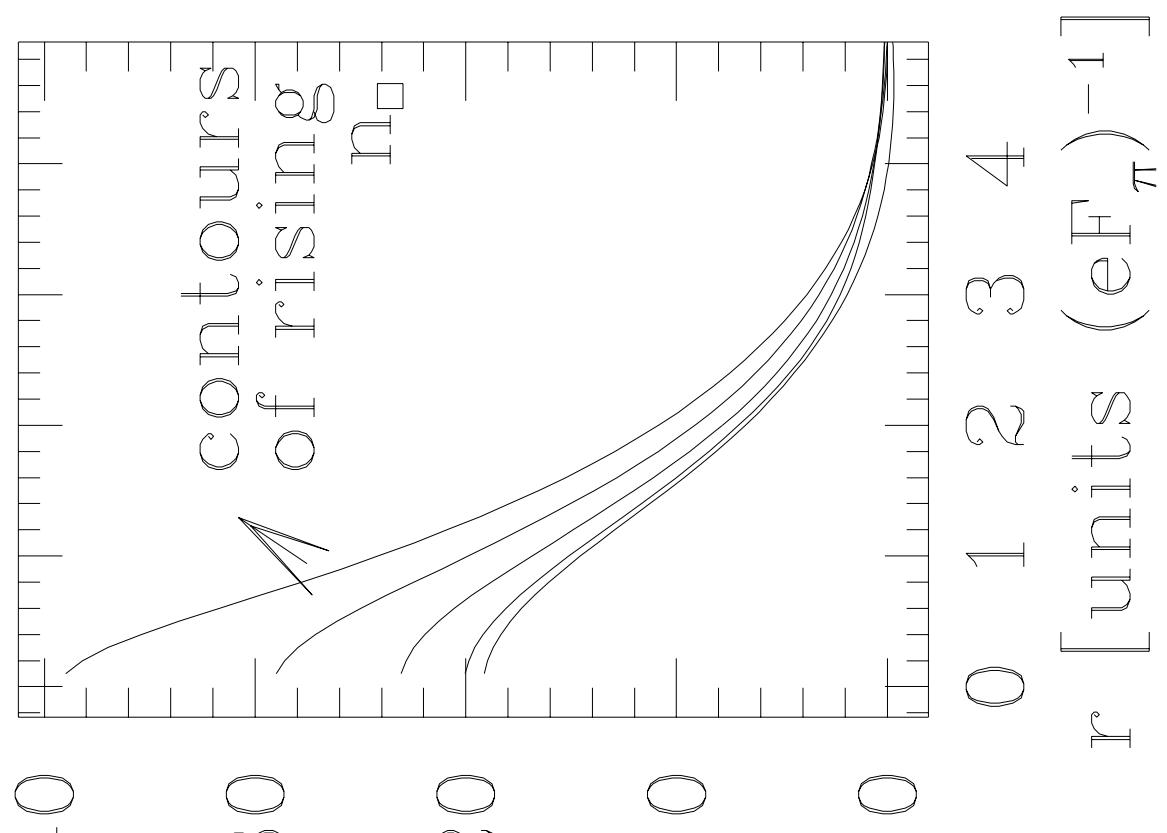
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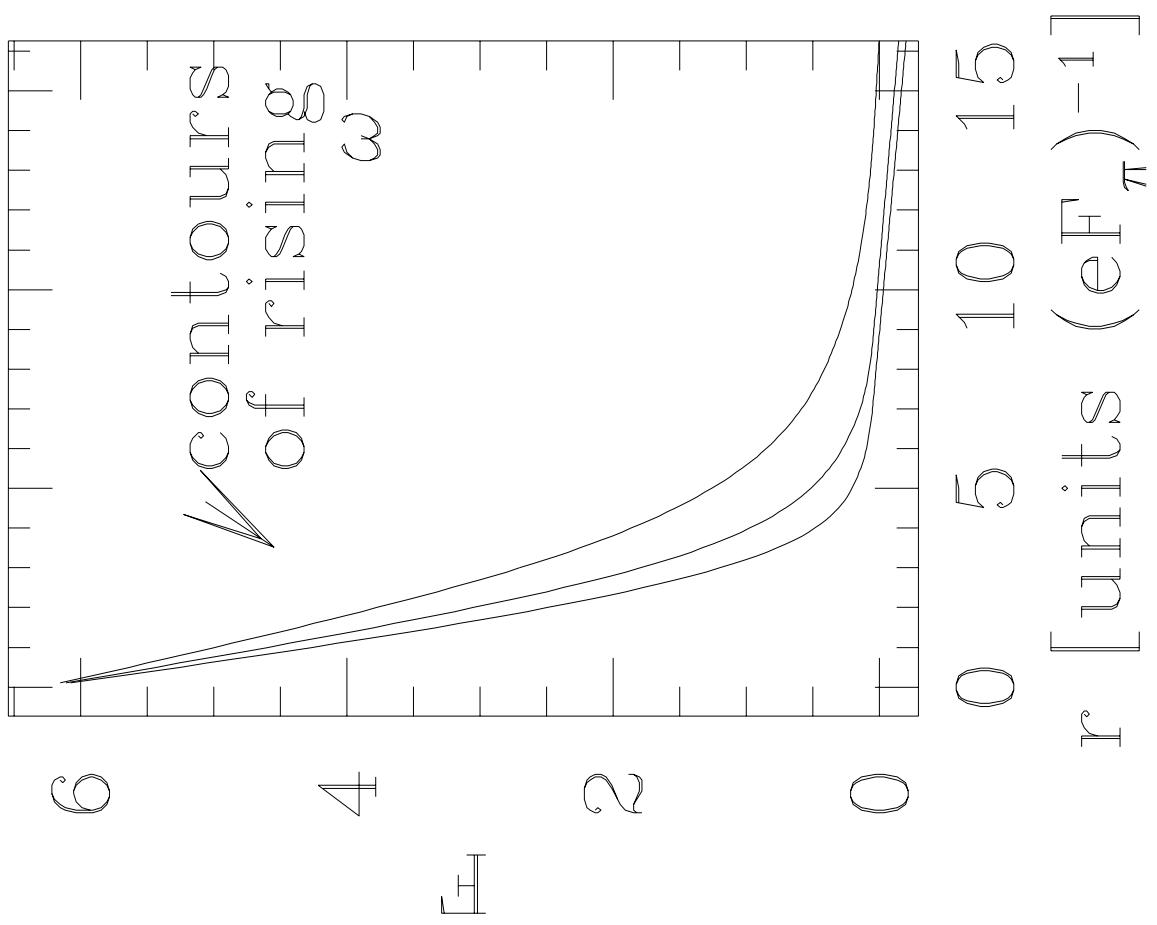
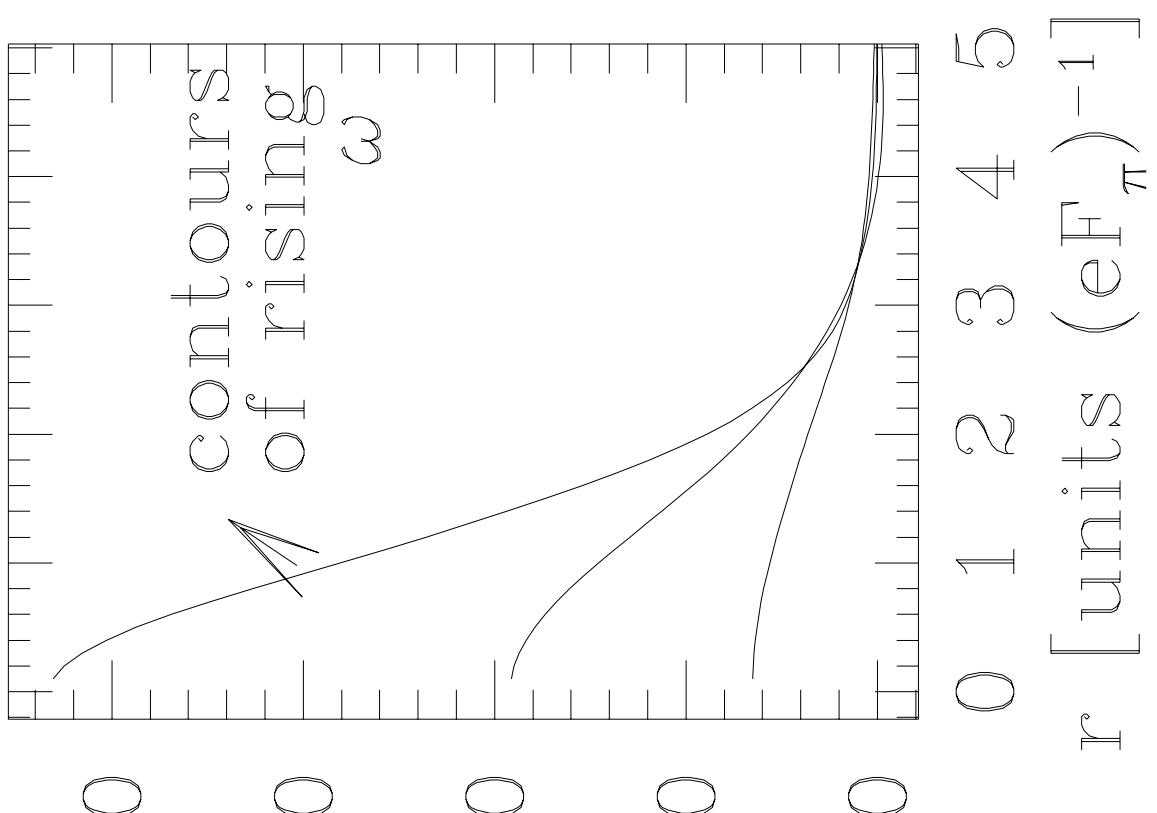
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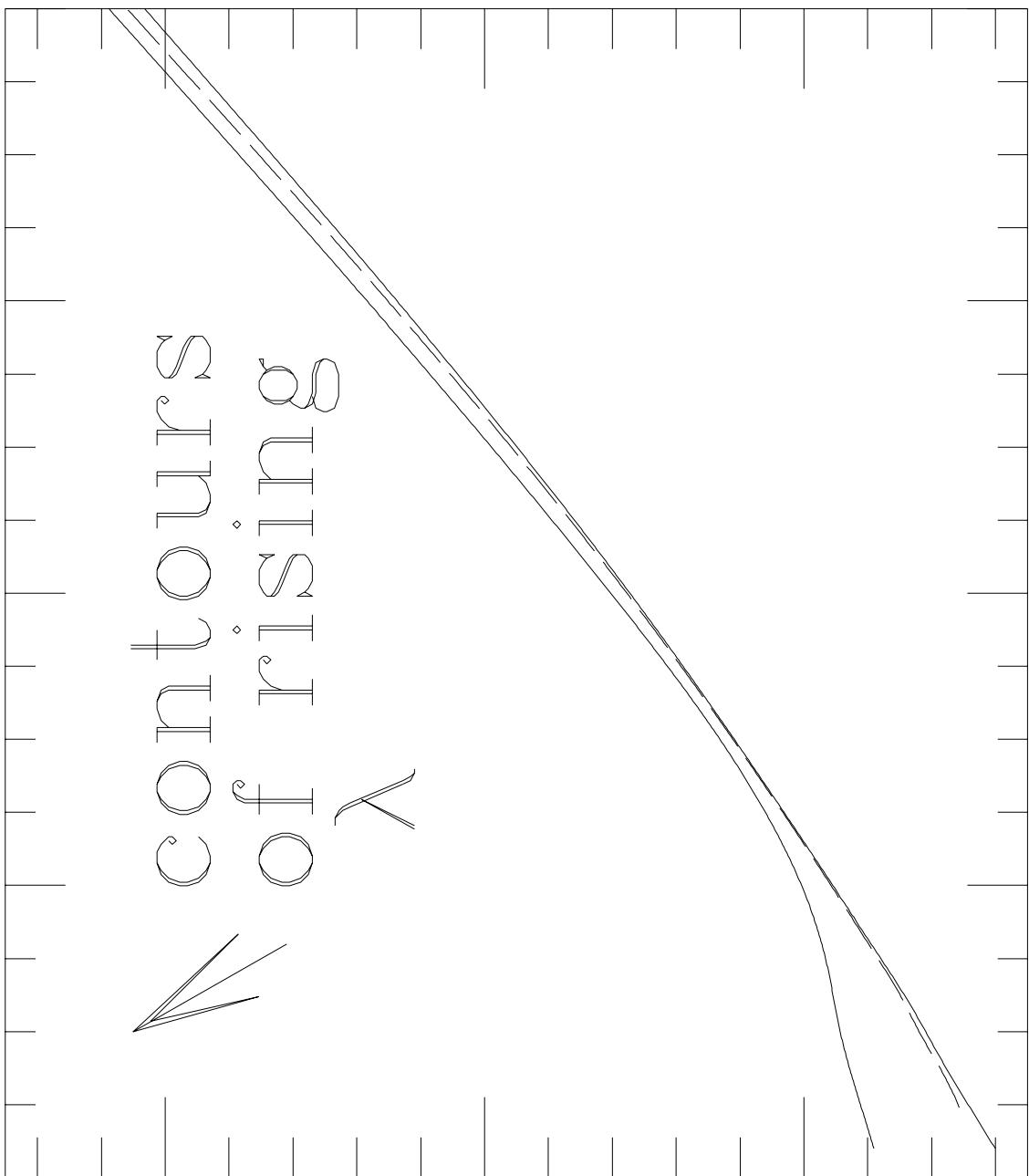
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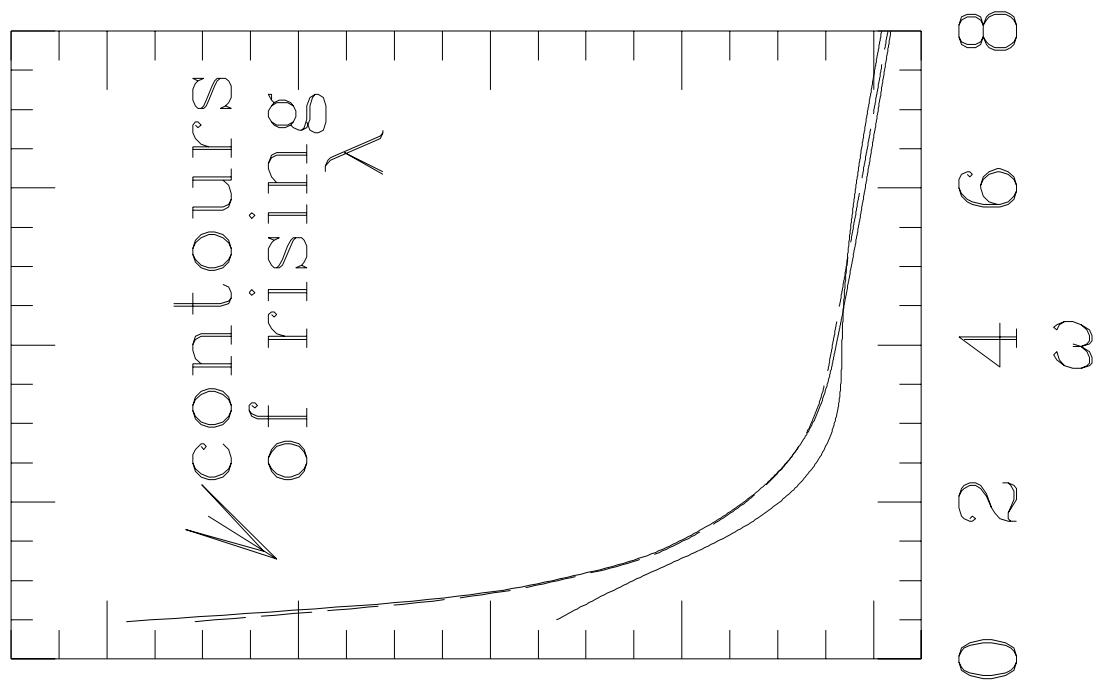
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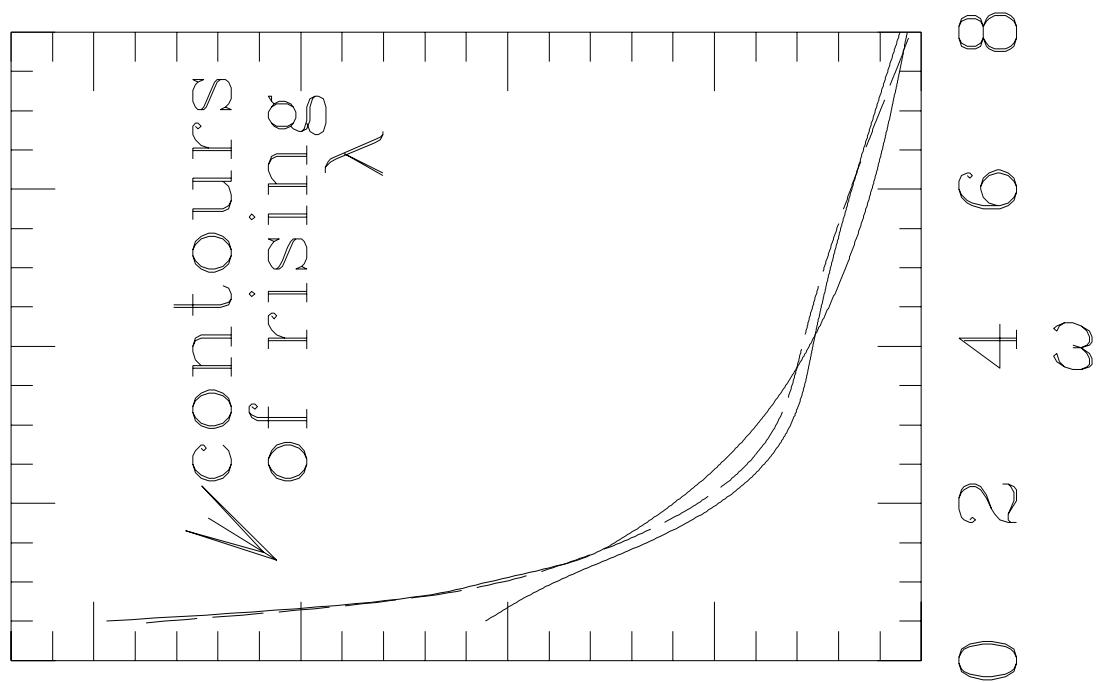


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